TD 3: P-RAM 2

1 Acceleration factor

Question 1 (Amdahl law)**.** Consider an algorithm with a percentage f of intrinsically sequential operations. Show that the acceleration factor is bounded by $1/f$, no matter how many processors. What to make of it?

Question 2 (Gustafson factor)**.** Gustafson introduced the acceleration factor $S_p = \frac{A(1)}{A(p)}$ where $A(p)$ is the average time of an arithmetic operation in a problem of the maximal size that can fit on p processors. Compute S_p for a $n \times n$ matrix operation with:

- n^α arithmetic operations,
- $w_1 n^2$ elements to be stored in memory,
- $w_2 n^2$ i/o (sequential) operations.

Question 3. Give examples of problems with super-linear acceleration factors.

2 Givens rotations on a linear network

To triangulate a matrix while staying numerically stable, we can use Givens rotations. The base operation is `rot(`i`,`k`)` that combines lines i and $i+1$, if they begin with $k-1$ zeros, to replace with 0 the element of coordinates $(i+1, k)$, for some magic θ:

$$\begin{pmatrix} 0 \dots 0 & a'_{i,k} & a'_{i,k+1} & \dots & a'_{i,n} \\ 0 \dots 0 & 0 & a'_{i+1,k+1} & \dots & a'_{i+1,n} \end{pmatrix} \leftarrow \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \dots 0 & a_{i,k} & a_{i,k+1} & \dots a_{i,n} \\ 0 \dots 0 & a_{i+1,k} & a_{i+1,k+1} & \dots a_{i+1,n} \end{pmatrix}$$

The sequential algorithm can be written:

```
for k = 1 to n-1:
    for i = n-1 downto k:
        rot(i,k)
```

Question 4. Distribute this algorithm on a linear network of n processors. (Do not assume that `rot(`i`,`k`)` gets faster when k increases.)

Question 5. Same question for only $n/2$ processors.

3 Connected components on P-RAMs

We want a CREW algorithm that computes the connected components of a graph (V, E). More precisely if $V = [\![1, n]\!]$, we want an array C of size n such that $C(i) = C(j) = k$ if and only i and j are in the same connected component and k is the smallest node of this component.

Definition 1. At each step of the algorithm, we call pseudo-node labelled with i the set of nodes $C^{-1}(i) = \{j \mid C(j) = i\}$. We will sometimes identify i and $C^{-1}(i)$.

The main invariant of the algorithm is that the smallest node of $C^{-1}(i)$ is i and that the nodes inside the same pseudo-node are in the same connected component. This assertion holds if we initialise C with $\forall i \in V \quad C(i) = i$, meaning that each node considers itself the reference node for its connected component. The goal of the algorithm is to change this selfish attitude.

Definition 2. A *k-cyclic pseudo-tree* is a directed graph weakly connected (i.e. the induced undirected graph is connected) such that:

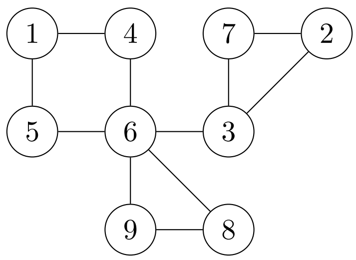
- the outdegree of every node is 1,
- there is exactly one cycle of length $k + 1$.

A *star* is a 0-cyclic pseudo-tree where all edges point to the smallest node (the *root*).

The invariant above is that the directed graph $G_C = (V, \{(i, C(i)) \mid i \in V\})$ is made of stars. We identify pseudo-nodes and stars, the centre of the stars being the index of the pseudo-node. The computation of connected components is done by iterating the following two functions:

```
gather():
    for i∈V in parallel do:
        N(i) := min{C(j) | {i,j}∈E, C(i)≠C(j)} or C(i) if the set is empty
    for i∈V in parallel do:
        T(i) := min{N(j) | C(j)=i, N(j)≠i} or C(i) if the set is empty
jump():
    for i∈V in parallel do:
        B(i) := T(i)
    repeat log(n) times:
        for i∈V in parallel do:
            T(i) := T(T(i))
    for i∈V in parallel do:
        C(i) := min{(B(T(i)),T(i)}
```

Question 6. Apply `gather` on the following graph, then `jump`, then `gather`, etc. Keep track of the graphs G_C and $G_T = (V, \{(i, T(i)) \mid i \in V\})$.



Question 7. Show that after an application of `gather`, connected components containing several pseudo-nodes induce 1-cyclic pseudo-trees in G_T. Note that the smallest pseudo-node of a 1-cyclic pseudo-tree is inside the cycle.

Question 8. Show that `jump` transforms a 1-cyclic pseudo-tree into a star.

Question 9. Show that after $\lceil \log n \rceil$ combinations of `gather` and `jump` the connected components of the graph are represented by C.

Question 10. What is the complexity of the algorithm? How many processors are used?