Basics of Deductive Program Verification

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Cours MPRI 2-36-1 “Preuve de Programme”

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Preliminaries

Very first question

Lectures in English or in French?

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- Schedule on the Web page `https://marche.gitlabpages.inria.fr/lecture-deductive-verif/`
- Lectures 1,2,3,4: Claude Marché
- Lectures 5,6,7,8: Jean-Marie Madiot
- (to confirm) some lecture could be replaced by practical lab, support for project. Project due date around mid-February.

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- Evaluation:
  - project P using the Why3 tool (http://why3.lri.fr)
  - final exam E: date to decide
  - final mark = $(2E + P + max(E, P))/4$
- internships (*stages*)

Outline

General Objectives

Ultimate Goal

Verify that software is free of bugs

Famous software failures:

http://www.cs.tau.ac.il/~nachumd/horror.html

This lecture

Computer-assisted approaches for verifying that
a software conforms to a specification

Some general approaches to Verification

Static analysis, Algorithmic Verification

- *model checking* (automata-based models)
- *abstract interpretation* (domain-specific model, e.g. numerical)

Deductive verification

- formal models using expressive logics
- verification = computer-assisted mathematical proof

Some general approaches to Verification

Refinement

- Formal models
- Code derived from model, correct by construction

A long time before success

Computer-assisted verification is an old idea

- Turing, 1948
- Floyd-Hoare logic, 1969

Success in practice: only from the mid-1990s

- Importance of the *increase of performance of computers*

A first success story:

- Paris metro line 14, using *Atelier B* (1998, refinement approach)

Other Famous Success Stories

- Flight control software of A380: *Astree* verifies absence of run-time errors (2005, abstract interpretation)
  `http://www.astree.ens.fr/`
- Microsoft's hypervisor: using Microsoft's *VCC* and the *Z3* automated prover (2008, deductive verification)
  `http://research.microsoft.com/en-us/projects/vcc/`
  More recently: verification of PikeOS
- Certified C compiler, developed using the *Coq* proof assistant (2009, correct-by-construction code generated by a proof assistant)
  `http://compcert.inria.fr/`
- L4.verified micro-kernel, using tools on top of *Isabelle/HOL* proof assistant (2010, Haskell prototype, C code, proof assistant)
  `http://www.ertos.nicta.com.au/research/l4.verified/`

Other Success Stories at Industry

- Frama-C
  - EDF: abstract interpretation
  - Airbus: deductive verification
- Spark/Ada: Verification of Ada programs

  https://www.adacore.com/industries

Remark

The two above use Why3 internally

Outline

Proposition logic in a nutshell

- Syntax:

$$\begin{array}{lcll} \varphi & ::= & \bot \mid \top \mid A, B & \text{(atoms)} \\ & \mid & \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \neg\varphi & \\ & \mid & \varphi \rightarrow \varphi \mid \varphi \leftrightarrow \varphi & \end{array}$$

- Semantics, models: truth tables

ϕ is satisfiable if it has a model
ϕ is valid if true in all models
(equivalently $\neg\phi$ is not satisfiable)

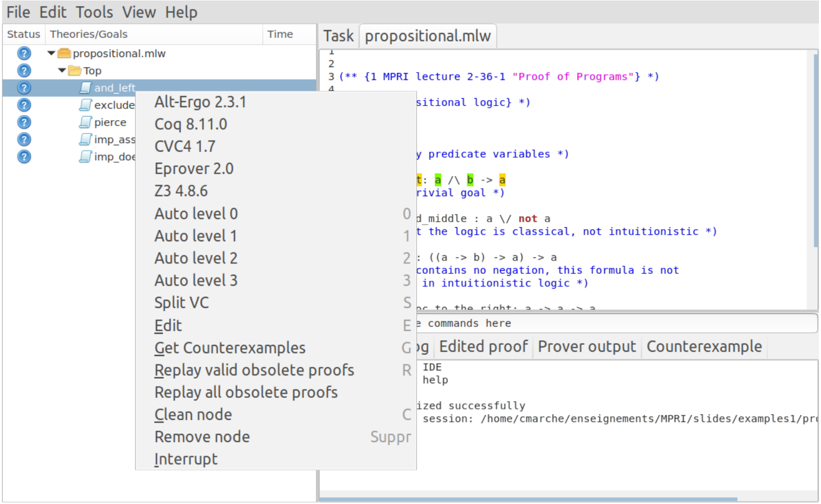
SAT is *decidable* $\rightsquigarrow$ SAT solvers

Demo with Why3

```
$ why3 ide propositional.mlw
```

Notice that Why3 indeed queries solvers for satisfiability of $\neg\phi$

Focus on the “Tools” menu of Why3



First-order logic in a nutshell

- Syntax:

$$
\begin{array}{rcll}
\varphi & ::= & \cdots & \\
 & | & P(t,\ldots,t) & \text{(predicates)} \\
 & | & \forall x.\ \phi \mid \exists x.\ \phi & \\
t & ::= & x & \text{variables} \\
 & | & f(t,\ldots,t) & \text{(function symbols)}
\end{array}
$$

- Semantics: models must interpret variables. C
- Satisfiability *undecidable*, but still *semi-decidable*: there exists complete systems of deduction rules (sequent calculus, natural deduction, superposition calculus)
- Examples of solvers: E, Spass, Vampire

Implement *refutationally complete* procedure:
if they answer 'unsat' then formula is unsatisfiable

Demo with Why3

`first-order.mlw`

Notice that Why3 logic is *typed*, and application is curryied

Logic Theories

- *Theory* = set of formulas (called *theorems*) closed by logical consequence
- *Axiomatic Theory* = set of formulas generated by axioms (or axiom schemas)
- *Consistent Theory*

  for no P, P and $\neg P$ are both theorems
  equivalently: 'false' is not a theorem
  equivalently: the theory has models

- *Consistent Axiomatization*
  'false' is not derivable

Theory of Equality

$$\forall x.\ x = x$$

$$\forall x, y.\ x = y \rightarrow y = x$$

$$\forall x, y, z.\ x = y \land y = z \rightarrow x = z$$

(congruence) for all function symbols f of arity k:

$$\forall x_1, y_1 \ldots, x_k, y_k.\ x_1 = y_1 \land \cdots \land x_k = y_k \rightarrow f(x_1, \ldots, x_k) = f(y_1, \ldots, y_k)$$

and for all predicates p of arity k:

$$\forall x_1, y_1 \ldots, x_k, y_k.\ x_1 = y_1 \land \cdots \land x_k = y_k \rightarrow p(x_1, \ldots, x_k) \rightarrow p(y_1, \ldots, y_k)$$

Theory of Equality, Continued

$$\forall x.\ x = x$$

$$\forall x, y.\ x = y \rightarrow y = x$$

$$\forall x, y, z.\ x = y \land y = z \rightarrow x = z$$

(congruence) ...

- General first-order deduction bad in such a case $\rightsquigarrow$ dedicated methods
  - paramodulation
  - congruence closure (for quantifier-free fragment)
- SMT solvers (Alt-Ergo, CVC4, Z3) implement dedicated (semi-)decision procedures

Demo with Why3

`equality.mlw`

Theories Continued

Theory of a given model

= formulas true in this model

- Central example: theory of linear integer arithmetic, i.e. formulas using $+$ and $\leq$
  - First-order theory is known to be decidable (Presburger)
  - SMT solvers typically implement a procedure for the existential fragment
- Also: theory of (non-linear) real arithmetic is decidable (Tarski)

Non-linear Integer Arithmetic

(a.k.a. Peano Arithmetic)

First-Order Integer Arithmetic

All valid first-order formulas on integers with $+$, $\times$ and $\leq$

- This theory is not even semi-decidable
- SMT solvers implement incomplete satisfiability checks: if solver answers 'unsat' then it is unsatisfiable

Demo with Why3

`arith.mlw`

Digression about Non-linear Integer Arithmetic

Representation Theorem (Gödel)

Every recursive function f is representable by a predicate ϕ_f such that

$$\phi_f(x_1, \ldots, x_k, y)$$

is true if and only if

$$y = f(x_1, \ldots, x_k)$$

First incompleteness Theorem (Gödel)

That theory is not recursively axiomatizable

Summary of prover limitations

- Superposition solvers (E, Spass,)
  - do not support well theories except equality
  - do quite well with quantifiers
- SMT solvers (Alt-Ergo, CVC4, Z3)
  - several theories are built-in
  - weaker with quantifiers
- None support reasoning by induction

Outline

IMP language

IMP language

A very basic imperative programming language

- only global variables
- only integer values for expressions
- basic statements:
  - assignment x `<-` e
  - sequence s_1 ; s_2
  - conditionals `if` e `then` s_1 `else` s_2
  - loops `while` e `do` s
  - no-op `skip`

Formal Contracts

General form of a program:

Contract

- *precondition*: expresses what is assumed before running the program
- *post-condition*: expresses what is supposed to hold when program exits

Demo with Why3

`contracts.mlw`

Hoare triples

- *Hoare triple* : notation $\{P\}s\{Q\}$
- P : formula called the *precondition*
- Q : formula called the *postcondition*

Intended meaning

$\{P\}s\{Q\}$ is true if and only if:
when the program s is executed in any state satisfying P, then (if execution terminates) its resulting state satisfies Q

This is a *Partial Correctness*: we say nothing if s does not terminate

Examples

Examples of valid triples for partial correctness:

- $\{x = 1\} x$ <- $x + 2 \{x = 3\}$
- $\{x = y\} x$ <- $x + y \{x = 2 * y\}$
- $\{\exists v.\ x = 4 * v\} x$ <- $x + 42 \{\exists w.\ x = 2 * w\}$
- $\{\mathit{true}\}$`while` 1 `do skip`$\{\mathit{false}\}$

Running Example

Three global variables `n`, `count`, and `sum`

```
count <- 0; sum <- 1;
while sum ≤ n do
  count <- count + 1; sum <- sum + 2 ∗ count + 1
```

Running Example

Three global variables `n`, `count`, and `sum`

```
count <- 0; sum <- 1;
while sum ≤ n do
  count <- count + 1; sum <- sum + 2 * count + 1
```

What does this program compute?

(assuming input is `n` and output is `count`)

Running Example

Three global variables `n`, `count`, and `sum`

```
count <- 0; sum <- 1;
while sum ≤ n do
  count <- count + 1; sum <- sum + 2 * count + 1
```

What does this program compute?

(assuming input is `n` and output is `count`)

Informal specification:

- at the end of execution of this program, `count` contains the square root of `n`, rounded downward
- e.g. for `n=42`, the final value of `count` is 6.

See file `imp_isqrt.mlw`

Hoare logic as an Axiomatic Semantics

Original Hoare logic *[∼ 1970]*

Axiomatic Semantics of programs

Set of *inference rules* producing triples

$$\frac{}{\{P\}\texttt{skip}\{P\}}$$

$$\frac{}{\{P[x \leftarrow e]\}x \texttt{<-} e\{P\}}$$

$$\frac{\{P\}s_1\{Q\} \qquad \{Q\}s_2\{R\}}{\{P\}s_1; s_2\{R\}}$$

- Notation $P[x \leftarrow e]$: replace all occurrences of program variable x by e in P.

Hoare Logic, continued

Frame rule:

$$\frac{\{P\}s\{Q\}}{\{P \land R\}s\{Q \land R\}}$$

with R a formula where no variables assigned in s occur

Consequence rule:

$$\frac{\{P'\}s\{Q'\} \qquad \models P \to P' \qquad \models Q' \to Q}{\{P\}s\{Q\}}$$

- Example: proof of

$$\{x = 1\}x \texttt{ <- } x + 2\{x = 3\}$$

Hoare Logic, continued

Rules for if and while :

$$\frac{\{P \land e\}s_1\{Q\} \qquad \{P \land \neg e\}s_2\{Q\}}{\{P\}\texttt{if } e \texttt{ then } s_1 \texttt{ else } s_2\{Q\}}$$

$$\frac{\{I \land e\}s\{I\}}{\{I\}\texttt{while } e \texttt{ do } s\{I \land \neg e\}}$$

I is called a *loop invariant*.

Example: isqrt(42)

Exercise: prove of the triple

$$\{n \geq 0\}\ \mathit{ISQRT}\ \{count^2 \leq n \wedge n < (count + 1)^2\}$$

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Could we do that by hand?

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Back to demo: file `imp_isqrt.mlw`

Example: isqrt(42)

Exercise: prove of the triple

$$\{n \geq 0\}\ \mathit{ISQRT}\ \{\mathit{count}^2 \leq n \land n < (\mathit{count} + 1)^2\}$$

Could we do that by hand?

Back to demo: file `imp_isqrt.mlw`

Warning

Finding an adequate loop invariant is a major difficulty

Beyond Axiomatic Semantics

- Operational Semantics

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- Semantic Validity of Hoare Triples

Beyond Axiomatic Semantics

- Operational Semantics
- Semantic Validity of Hoare Triples
- Hoare logic as correct deduction rules

Operational semantics

[Plotkin 1981, structural operational semantics (SOS)]

- we use a standard *small-step semantics*
- *program state*: describes content of global variables at a given time. It is a finite map Σ associating to each variable x its current value denoted $\Sigma(x)$.
- Value of an expression e in some state Σ:
  - denoted $[\![e]\!]_\Sigma$
  - always defined, by the following recursive equations:

$$\begin{array}{rcl} [\![n]\!]_\Sigma & = & n \\ [\![x]\!]_\Sigma & = & \Sigma(x) \\ [\![e_1\ op\ e_2]\!]_\Sigma & = & [\![e_1]\!]_\Sigma\ [\![op]\!]\ [\![e_2]\!]_\Sigma \end{array}$$

  - $[\![op]\!]$ natural semantic of operator op on integers (with relational operators returning 0 for false and $\neq 0$ for true).

Semantics of statements

Semantics of statements: defined by judgment

$$\Sigma, s \quad \rightsquigarrow \quad \Sigma', s'$$

meaning: in state Σ, executing one step of statement s leads to the state Σ' and the remaining statement to execute is s'.
The semantics is defined by the following rules.

$$\frac{}{\Sigma, x \texttt{ <- } e \rightsquigarrow \Sigma\{x \leftarrow [\![e]\!]_\Sigma\}, \texttt{skip}}$$

$$\frac{\Sigma, s_1 \rightsquigarrow \Sigma', s_1'}{\Sigma, (s_1; s_2) \rightsquigarrow \Sigma', (s_1'; s_2)}$$

$$\frac{}{\Sigma, (\texttt{skip}; s) \rightsquigarrow \Sigma, s}$$

Semantics of statements, continued

$$\frac{[\![e]\!]_\Sigma \neq 0}{\Sigma, \texttt{if } e \texttt{ then } s_1 \texttt{ else } s_2 \rightsquigarrow \Sigma, s_1}$$

$$\frac{[\![e]\!]_\Sigma = 0}{\Sigma, \texttt{if } e \texttt{ then } s_1 \texttt{ else } s_2 \rightsquigarrow \Sigma, s_2}$$

$$\frac{[\![e]\!]_\Sigma \neq 0}{\Sigma, \texttt{while } e \texttt{ do } s \rightsquigarrow \Sigma, (s; \texttt{while } e \texttt{ do } s)}$$

$$\frac{[\![e]\!]_\Sigma = 0}{\Sigma, \texttt{while } e \texttt{ do } s \rightsquigarrow \Sigma, \texttt{skip}}$$

Execution of programs

- $\rightsquigarrow$: a binary relation over pairs (state,statement)
- transitive closure : $\rightsquigarrow^+$
- reflexive-transitive closure : $\rightsquigarrow^*$

In other words:

$$\Sigma, s \overset{*}{\rightsquigarrow} \Sigma', s'$$

means that statement s, in state Σ, reaches state Σ' with remaining statement s' after executing some finite number of steps.

Running example:

$$\{n = 42, count = ?, sum = ?\}, ISQRT \rightsquigarrow^* \\ \{n = 42, count = 6, sum = 49\}, \texttt{skip}$$

Execution and termination

- any statement except `skip` can execute in any state
- the statement `skip` alone represents the final step of execution of a program
- there is no possible *runtime error*.

Definition

Execution of statement s in state Σ *terminates* if there is a state Σ' such that $\Sigma, s \rightsquigarrow^* \Sigma', \texttt{skip}$

- since there are no possible runtime errors, s does not terminate means that s *diverges* (i.e. executes infinitely).

Semantics of formulas

$[\![p]\!]_\Sigma$:

- semantics of formula p in program state Σ
- is a logic formula where no program variables appear anymore
- defined recursively as follows.

$$\begin{array}{rcl} [\![e]\!]_\Sigma & = & [\![e]\!]_\Sigma \neq 0 \\ [\![p_1 \ \land \ p_2]\!]_\Sigma & = & [\![p_1]\!]_\Sigma \ \land \ [\![p_2]\!]_\Sigma \\ & \vdots & \end{array}$$

where semantics of expressions is augmented with

$$\begin{array}{rcl} [\![v]\!]_\Sigma & = & v \\ [\![x]\!]_\Sigma & = & \Sigma(x) \end{array}$$

Notations:

- $\Sigma \models p$: the formula $[\![p]\!]_\Sigma$ is *valid*
- $\models p$: formula $[\![p]\!]_\Sigma$ holds in all states Σ.

Semantics of formulas

Other presentation of the semantics: $[\![p]\!]_\Sigma$:

- inline semantic of first-order formula
- $[\![e]\!]_{\Sigma,\mathcal{V}}$ with $\mathcal{V}$ mapping of logic variables to integers.
- defined recursively as follows.

$$
\begin{array}{rcl}
[\![p_1 \ \wedge \ p_2]\!]_{\Sigma,\mathcal{V}} & = & \begin{cases} \top & \text{if } [\![p_1]\!]_{\Sigma,\mathcal{V}} = \top \text{ and } [\![p_2]\!]_{\Sigma,\mathcal{V}} = \top \\ \bot & \end{cases} \\
[\![\forall x.e]\!] & = & \top \text{ if for all } v.\ [\![e]\!]_{\Sigma,\mathcal{V}[x\leftarrow v]} = \top \\
 & \vdots &
\end{array}
$$

where semantics of expressions is augmented with

$$
\begin{array}{rcl}
[\![v]\!]_{\Sigma,\mathcal{V}} & = & \mathcal{V}(v) \\
[\![x]\!]_{\Sigma,\mathcal{V}} & = & \Sigma(x)
\end{array}
$$

Soundness

Definition (Partial correctness)

Hoare triple $\{P\}s\{Q\}$ is said *valid* if:
for any states Σ, Σ', if

- $\Sigma, s \leadsto^* \Sigma', \texttt{skip}$ and
- $\Sigma \models P$

then $\Sigma' \models Q$

Theorem (Soundness of Hoare logic)

The set of rules is correct: any derivable triple is valid.

This is *proved by induction on the derivation tree* of the considered triple.
For each rule: assuming that the triples in premises are valid, we show that the triple in conclusion is valid too.

Annotated Programs

Goal

Add automation to the Hoare logic approach

We augment IMP with *explicit loop invariants*

`while` e `invariant` I `do` s

Weakest liberal precondition

[Dijkstra 1975]

Function $\mathrm{WLP}(s, Q)$:

- s is a statement
- Q is a formula
- returns a formula

It should return the *minimal precondition* P that validates the triple $\{P\}s\{Q\}$

Definition of $\mathrm{WLP}(s, Q)$

Recursive definition:

$$
\begin{array}{rcl}
\mathrm{WLP}(\texttt{skip}, Q) & = & Q \\
\mathrm{WLP}(x \texttt{ <- } e, Q) & = & Q[x \leftarrow e] \\
\mathrm{WLP}(s_1; s_2, Q) & = & \mathrm{WLP}(s_1, \mathrm{WLP}(s_2, Q)) \\
\mathrm{WLP}(\texttt{if } e \texttt{ then } s_1 \texttt{ else } s_2, Q) & = & \\
(e \rightarrow \mathrm{WLP}(s_1, Q)) & \wedge & (\neg e \rightarrow \mathrm{WLP}(s_2, Q))
\end{array}
$$

Definition of $\mathrm{WLP}(s, Q)$, continued

$$
\begin{array}{ll}
\mathrm{WLP}(\texttt{while}\ e\ \texttt{invariant}\ I\ \texttt{do}\ s, Q) = & \\
\quad I \wedge & \text{(invariant true initially)} \\
\quad \forall v_1, \ldots, v_k. & \\
\qquad (((e \wedge I) \rightarrow \mathrm{WLP}(s, I)) & \text{(invariant preserved)} \\
\qquad \wedge((\neg e \wedge I) \rightarrow Q))[w_i \leftarrow v_i] & \text{(invariant implies post)}
\end{array}
$$

where $w_1, \ldots, w_k$ is the set of assigned variables in statement s and $v_1, \ldots, v_k$ are fresh logic variables

Examples

$$\mathrm{WLP}(x \text{ <- } x + y, x = 2y) \equiv x + y = 2y$$

Examples

$$\mathrm{WLP}(x \texttt{<-}\ x + y, x = 2y) \equiv x + y = 2y$$

$$\mathrm{WLP}(\texttt{while}\ y > 0\ \texttt{invariant}\ \mathit{even}(y)\ \texttt{do}\ y \texttt{<-}\ y - 2, \mathit{even}(y)) \equiv$$

Examples

$$\mathrm{WLP}(x \texttt{<-}\ x + y, x = 2y) \ \equiv\ x + y = 2y$$

$$\begin{aligned}
&\mathrm{WLP}(\texttt{while}\ y > 0\ \texttt{invariant}\ \mathit{even}(y)\ \texttt{do}\ y \texttt{<-}\ y - 2, \mathit{even}(y)) \ \equiv \\
&\quad \mathit{even}(y) \wedge \\
&\quad \forall v, ((v > 0 \wedge \mathit{even}(v)) \rightarrow \mathit{even}(v - 2)) \\
&\qquad \wedge ((v \leq 0 \wedge \mathit{even}(v)) \rightarrow \mathit{even}(v))
\end{aligned}$$

Soundness

Theorem (Soundness)

For all statement s and formula Q, $\{\mathrm{WLP}(s, Q)\}s\{Q\}$ is valid.

Proof by induction on the structure of statement s.

Consequence

For proving that a triple $\{P\}s\{Q\}$ is valid, it suffices to prove the formula $P \rightarrow \mathrm{WLP}(s, Q)$.

This is basically the goal that Why3 produces

Digression: Completeness of Hoare Logic

Two major difficulties for proving a program

- *guess the appropriate intermediate formulas* (for sequence, for the loop invariant)
- *prove the logical premises of consequence rule*

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Theoretical question: completeness. Are all valid triples derivable from the rules?

Theorem (Relative Completeness of Hoare logic)

The set of rules of Hoare logic is relatively complete: if the logic language is expressive enough, then any valid triple $\{P\}s\{Q\}$ can be derived using the rules.

Digression: Completeness of Hoare Logic

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- *guess the appropriate intermediate formulas* (for sequence, for the loop invariant)
- *prove the logical premises of consequence rule*

Theoretical question: completeness. Are all valid triples derivable from the rules?

Theorem (Relative Completeness of Hoare logic)

The set of rules of Hoare logic is relatively complete: if the logic language is expressive enough, then any valid triple $\{P\}s\{Q\}$ can be derived using the rules.

[Cook, 1978] "Expressive enough": representability of any recursive function

Yet, this does not provide an effective recipe to discover suitable loop invariants (see also the theory of abstract interpretation *[Cousot, 1990]*)

Outline

Exercise 1

Consider the following (inefficient) program for computing the sum $a + b$.

```
x <- a; y <- b;
while y > 0 do
   x <- x + 1; y <- y - 1
```

(Why3 file to fill in: `imp_sum.mlw`)

- Propose a post-condition stating that the final value of x is the sum of the values of a and b
- Find an appropriate loop invariant
- Prove the program.

Exercise 2

The following program is one of the original examples of Floyd.

```
q <- 0; r <- x;
while r ≥ y do
   r <- r - y; q <- q + 1
```

(Why3 file to fill in: `imp_euclide.mlw`)

- Propose a formal precondition to express that x is assumed non-negative, y is assumed positive, and a formal post-condition expressing that q and r are respectively the quotient and the remainder of the Euclidean division of x by y.
- Find appropriate loop invariant and prove the correctness of the program.

Exercise 3

Let's assume given in the underlying logic the functions div2(x) and mod2(x) which respectively return the division of x by 2 and its remainder. The following program is supposed to compute, in variable r, the power x^n.

```
r ≤ 1; p <- x; e <- n;
while e > 0 do
  if mod2(e) ≠ 0 then r <- r * p;
  p <- p * p;
  e <- div2(e);
```

(Why3 file to fill in: `power_int.mlw`)

- Assuming that the power function exists in the logic, specify appropriate pre- and post-conditions for this program.
- Find an appropriate loop invariant, and prove the program.

Exercise 4

The Fibonacci sequence is defined recursively by $fib(0) = 0$, $fib(1) = 1$ and $fib(n + 2) = fib(n + 1) + fib(n)$. The following program is supposed to compute fib in linear time, the result being stored in y.

```
y <- 0; x <- 1; i <- 0;
while i < n do
   aux <- y; y <- x; x <- x + aux; i <- i + 1
```

- Assuming fib exists in the logic, specify appropriate pre- and post-conditions.
- Prove the program.

Exercise (Exam 2011-2012)

In this exercise, we consider the simple language of the first lecture of this course, where expressions do not have any side effect.

1. *Prove that the triple*

$$\{P\}x \texttt{ <- } e\{\exists v,\ e[x \leftarrow v] = x \land P[x \leftarrow v]\}$$

   is valid with respect to the operational semantics.

2. *Show that the triple above can be proved using the rules of Hoare logic.*

Let us assume that we replace the standard Hoare rule for assignment by the rule

$$\overline{\{P\}x \texttt{ <- } e\{\exists v,\ e[x \leftarrow v] = x \land P[x \leftarrow v]\}}$$

3. *Show that the triple* $\{P[x \leftarrow e]\}x \texttt{ <- } e\{P\}$ *can be proved with the new set of rules.*

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