Separation Logic 4/4

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slides (mostly) from Arthur Charguéraud

Separation Logic 4/4

Before we start:

- Don't check "listen only": activate your microphone!
- For the first few minutes: show me your faces (if you reasonably can)

Also, an ad for an internship:

A formal proof of correctness of tail-recursion modulo constructor in the Iris logic
with: Léo Stefanesco, François Pottier, Gabriel Scherer, Frédéric Bour
`https://www.stefanesco.com/documents/internship-trmc.pdf`

Chapter 18

Characteristic Formulas with structural rules

Integration of structural rules

$$
\begin{aligned}
[\![v]\!] &\equiv \mathsf{local}\,(\lambda HQ.\;\; H \rhd Q\,v) \\
[\![\mathsf{let}\,x = t_1\,\mathsf{in}\,t_2]\!] &\equiv \mathsf{local}\,(\lambda HQ.\;\; \exists Q'.\; [\![t_1]\!]\,H\,Q' \\
&\qquad\qquad\qquad\qquad \wedge \forall x.\; [\![t_2]\!]\,(Q'\,x)\,Q\;) \\
[\![\mathsf{if}\,b\,\mathsf{then}\,t_1\,\mathsf{else}\,t_2]\!] &\equiv \mathsf{local}\,(\lambda HQ.\;\; (b = \mathsf{true} \Rightarrow [\![t_1]\!]\,H\,Q) \\
&\qquad\qquad\qquad \wedge\;\; (b = \mathsf{false} \Rightarrow [\![t_2]\!]\,H\,Q)\;) \\
[\![v_1\,v_2]\!] &\equiv \mathsf{local}\,(\lambda HQ.\;\; \mathsf{App}\,v_1\,v_2\,H\,Q) \\
[\![\mathsf{let\,rec}\,f = \lambda x.\,t_1\,\mathsf{in}\,t_2]\!] &\equiv \mathsf{local}\,(\lambda HQ.\;\; \forall f.\; P f \Rightarrow [\![t_2]\!]\,H\,Q)
\end{aligned}
$$

where $P f \equiv (\forall x H' Q'.\, [\![t_1]\!]\,H'\,Q' \Rightarrow \mathsf{App}\,f\,x\,H'\,Q')$

Definition of the local predicate (1/2)

To support:

$$\frac{H = H_1 \star H_2 \qquad [\![t]\!]\, H_1\, Q_1 \qquad Q_1 \star H_2 = Q}{[\![t]\!]\, H\, Q}\ \textsc{frame}'$$

Definition of the local predicate (1/2)

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we would define:

$$\mathsf{local}\,\mathcal{F} \;\equiv\; \lambda HQ.\ \exists H_1 H_2 Q_1. \begin{cases} H = H_1 \star H_2 \\ \mathcal{F}\, H_1\, Q_1 \\ Q_1 \star H_2 = Q \end{cases}$$

Framing using the local predicate

To prove "$\{H\}\ t\ \{Q\}$", by the rule FRAME', it suffices to show:

$$H = H_1 \star H_2 \quad \wedge \quad \{H_1\}\ t\ \{Q_1\} \quad \wedge \quad Q_1 \star H_2 = Q$$

To prove "local $[\![t]\!]\ H\ Q$", by definition of "local", it suffices to show:

$$H = H_1 \star H_2 \quad \wedge \quad [\![t]\!]\ H_1\ Q_1 \quad \wedge \quad Q_1 \star H_2 = Q$$

Definition of the local predicate (2/2)

To support:

$$\frac{H \rhd H_1 \star H_2 \qquad \{H_1\}\ t\ \{Q_1\} \qquad Q_1 \star H_2 \rhd Q \star \mathsf{GC}}{\{H\}\ t\ \{Q\}}\ \textsc{combined}$$

$$\frac{\forall x.\ \{H\}\ t\ \{Q\}}{\{\exists x.\, H\}\ t\ \{Q\}}\ \textsc{exists} \qquad\qquad \frac{P\ \Rightarrow\ \{H\}\ t\ \{Q\}}{\{[P] \star H\}\ t\ \{Q\}}\ \textsc{prop}$$

Definition of the local predicate (2/2)

To support:

$$\frac{H \rhd H_1 \star H_2 \qquad \{H_1\}\ t\ \{Q_1\} \qquad Q_1 \star H_2 \rhd Q \star \mathsf{GC}}{\{H\}\ t\ \{Q\}}\ \text{COMBINED}$$

$$\frac{\forall x.\ \{H\}\ t\ \{Q\}}{\{\exists x.\ H\}\ t\ \{Q\}}\ \text{EXISTS} \qquad\qquad \frac{P\ \Rightarrow\ \{H\}\ t\ \{Q\}}{\{[P] \star H\}\ t\ \{Q\}}\ \text{PROP}$$

we define:

$$\mathsf{local}\,\mathcal{F}\ \equiv\ \lambda HQ.\ \forall h.\ H\,h \Rightarrow \exists H_1 H_2 Q_1. \begin{cases} (H_1 \star H_2)\,h \\ \mathcal{F}\,H_1\,Q_1 \\ Q_1 \star H_2 \rhd Q \star \mathsf{GC} \end{cases}$$

Iterated applications of structural rules

The local predicate may be duplicated as many times as needed:

$$\mathsf{local}\,[\![t]\!]\,H\,Q \;\;=\;\; \mathsf{local}\,(\mathsf{local}\,[\![t]\!])\,H\,Q$$

For example, to prove "$\mathsf{local}\,[\![t]\!]\,H\,Q$", it suffices to show:

$$H = H_1 \star H_2 \;\;\wedge\;\; \mathsf{local}\,[\![t]\!]\,H_1\,Q_1 \;\;\wedge\;\; Q_1 \star H_2 = Q$$

Iterated applications of structural rules

The local predicate may be duplicated as many times as needed:

$$\mathsf{local}\,[\![t]\!]\,H\,Q \;\;=\;\; \mathsf{local}\,(\mathsf{local}\,[\![t]\!])\,H\,Q$$

For example, to prove "$\mathsf{local}\,[\![t]\!]\,H\,Q$", it suffices to show:

$$H = H_1 \star H_2 \;\;\wedge\;\; \mathsf{local}\,[\![t]\!]\,H_1\,Q_1 \;\;\wedge\;\; Q_1 \star H_2 = Q$$

When not needed, "local" may be simply erased:

$$[\![t]\!]\,H\,Q \;\;\Rightarrow\;\; \mathsf{local}\,[\![t]\!]\,H\,Q$$

Notation for characteristic formulas

$$[\![\mathsf{let}\, x = t_1 \,\mathsf{in}\, t_2]\!] \;\equiv\; \mathsf{local}\,(\lambda H Q.\;\; \exists Q'.\; [\![t_1]\!]\, H\, Q' \;\wedge\; \forall x.\; [\![t_2]\!]\,(Q'\, x)\, Q)$$

Definition of Coq notation:

$$(\mathsf{Let}\; x = \mathcal{F}_1 \;\mathsf{in}\; \mathcal{F}_2) \;\equiv\; \mathsf{local}\,(\lambda H Q.\;\; \exists Q'.\; \mathcal{F}_1\, H\, Q' \;\wedge\; \forall x.\; \mathcal{F}_2\,(Q'\, x)\, Q)$$

Notation for characteristic formulas

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With this notation:

$$[\![\mathsf{let}\, x = t_1 \,\mathsf{in}\, t_2]\!] \;\equiv\; (\mathsf{Let}\; x = [\![t_1]\!] \;\mathsf{in}\; [\![t_2]\!])$$

Notation for characteristic formulas

$$[\![\mathsf{let}\, x = t_1 \,\mathsf{in}\, t_2]\!] \;\equiv\; \mathsf{local}\,(\lambda H Q.\;\; \exists Q'.\; [\![t_1]\!]\, H\, Q' \,\wedge\, \forall x.\; [\![t_2]\!]\,(Q'\,x)\,Q)$$

Definition of Coq notation:

$$(\mathsf{Let}\; x = \mathcal{F}_1 \;\mathsf{in}\; \mathcal{F}_2) \;\equiv\; \mathsf{local}\,(\lambda H Q.\;\; \exists Q'.\; \mathcal{F}_1\, H\, Q' \,\wedge\, \forall x.\; \mathcal{F}_2\,(Q'\,x)\,Q)$$

With this notation:

$$[\![\mathsf{let}\, x = t_1 \,\mathsf{in}\, t_2]\!] \;\equiv\; (\mathsf{Let}\; x = [\![t_1]\!] \;\mathsf{in}\; [\![t_2]\!])$$

Technically:

$$[\![\mathsf{let}\, x = t_1 \,\mathsf{in}\, t_2]\!] \;\equiv\; (\mathsf{Let}\; X = [\![t_1]\!] \;\mathsf{in}\; [\![([x \to X]\, t_2)]\!])$$

Characteristic formulas generation, with notation

$$
\begin{array}{lcl}
[\![v]\!] & \equiv & \mathsf{Ret}\ v \\
[\![\mathsf{let}\, x = t_1 \,\mathsf{in}\, t_2]\!] & \equiv & \mathsf{Let}\ x = [\![t_1]\!]\ \mathsf{in}\ [\![t_2]\!] \\
[\![\mathsf{if}\, b \,\mathsf{then}\, t_1 \,\mathsf{else}\, t_2]\!] & \equiv & \mathsf{If}\ b\ \mathsf{then}\ [\![t_1]\!]\ \mathsf{else}\ [\![t_2]\!] \\
[\![v_1\ v_2]\!] & \equiv & \mathsf{App}\, v_1\, v_2 \\
[\![\mathsf{let\,rec}\, f = \lambda x.\, t_1 \,\mathsf{in}\, t_2]\!] & \equiv & \mathsf{Let\,Rec}\, f\, x\ =\ [\![t_1]\!]\ \mathsf{in}\ [\![t_2]\!]
\end{array}
$$

Tactics for characteristic formulas

What the user sees:

$$\text{Let } x = \mathcal{F}_1 \text{ in } \mathcal{F}_2$$

What is hidden behind the notation:

$$\text{local}\,(\lambda H Q.\ \ \exists Q'.\ \mathcal{F}_1\, H\, Q' \ \wedge\ \forall x.\ \mathcal{F}_2\,(Q'\,x)\,Q)$$

What the user would need to execute:

```
apply local_erase; esplit; split.
```

Tactics for characteristic formulas

What the user sees:

$$\text{Let } x = \mathcal{F}_1 \text{ in } \mathcal{F}_2$$

What is hidden behind the notation:

$$\text{local}\,(\lambda H Q.\ \ \exists Q'.\ \mathcal{F}_1\, H\, Q' \ \wedge\ \forall x.\ \mathcal{F}_2\,(Q'\,x)\,Q)$$

What the user would need to execute:

```
apply local_erase; esplit; split.
```

What the user writes:

```
xlet.
```

Chapter 19

Higher-order representation predicates

Overview

1. Higher-order predicate:

$$p \leadsto \mathsf{MList}\, L \qquad \text{is generalized into} \qquad p \leadsto \mathsf{Mlistof}\, R\, L$$

2. Identity representation predicate:

$$p \leadsto \mathsf{Mlistof}\, \mathsf{Id}\, L \qquad \text{is the same as} \qquad p \leadsto \mathsf{MList}\, L$$

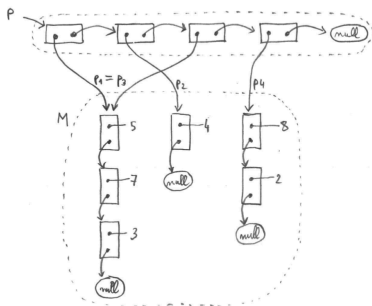
3. Control accesses:

$$\{p \leadsto \mathsf{MCellof}\, \mathsf{Id}\, v_1\, R_2\, V_2\}\ (p.\mathsf{hd})\ \{\lambda x.\, [x = v_1] \star \ldots\}$$

4. Compose recursively:

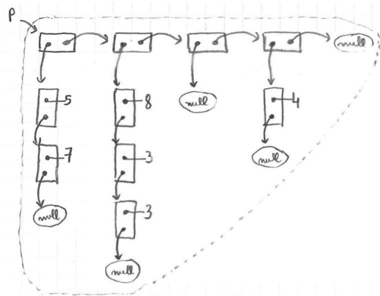
$$p \leadsto \mathsf{Nodeof}\, R\, X\, (\mathsf{Mlistof}\, (\mathsf{Narytreeof}\, R))\, L$$

Mutable list of possibly-aliased lists



$$p \rightsquigarrow \mathsf{MList}\, K \;\star\; \left(\underset{(p_i, L_i) \in M}{\circledast} p_i \rightsquigarrow \mathsf{MList}\, L_i \right) \;\star\; [\forall p_i \in K.\;\; p_i \in \mathsf{dom}\, M]$$

Mutable list of disjoint mutable lists

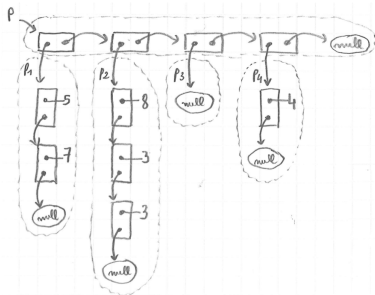


$$L \;=\; (5 :: 7 :: \mathsf{nil}) :: (8 :: 3 :: 3 :: \mathsf{nil}) \\ :: (\mathsf{nil}) :: (4 :: \mathsf{nil}) :: \mathsf{nil}$$

$$p \rightsquigarrow \mathsf{MlistofMlist}\, L$$

(to be later generalized into: $p \rightsquigarrow \mathsf{Mlistof}\, R\, L$)

Representation using iterated star

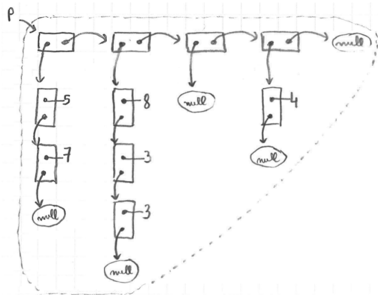


$$L = (5 :: 7 :: \mathsf{nil}) :: (8 :: 3 :: 3 :: \mathsf{nil}) :: (\mathsf{nil}) :: (4 :: \mathsf{nil}) :: \mathsf{nil}$$

$$K = p_1 :: p_2 :: p_3 :: p_4 :: \mathsf{nil}$$

$$p \rightsquigarrow \mathsf{MlistofMlist}\, L \quad \equiv \quad \exists K. \quad p \rightsquigarrow \mathsf{MList}\, K \;\star\; \circledast_{i \in [0, |L|)} (K[i]) \rightsquigarrow \mathsf{MList}\, (L[i]) \;\star\; [\, |K| = |L| \,]$$

Representation using a recursive predicate



$$L \;=\; (5 :: 7 :: \mathsf{nil}) :: (8 :: 3 :: 3 :: \mathsf{nil}) \\ :: (\mathsf{nil}) :: (4 :: \mathsf{nil}) \;::\; \mathsf{nil}$$

$$\begin{array}{rcl} p \leadsto \mathsf{MlistofMlist}\, L & \equiv & \mathsf{match}\, L\, \mathsf{with} \\ & & \mid \mathsf{nil} \;\Rightarrow\; [p = \mathsf{null}] \\ & & \mid X :: L' \;\Rightarrow\; \exists x p'.\;\; p \leadsto \{\!|\mathsf{hd}{=}x;\, \mathsf{tl}{=}p'|\!\} \\ & & \qquad\qquad\qquad\quad \star\; p' \leadsto \mathsf{MlistofMlist}\, L' \\ & & \qquad\qquad\qquad\quad \star\; x \leadsto \mathsf{MList}\, X \end{array}$$

Generalization to a higher-order predicate

$$p \rightsquigarrow \mathsf{MlistofMlist}\, L \;\equiv\; \mathsf{match}\, L\, \mathsf{with}$$
$$\mid \mathsf{nil} \;\Rightarrow\; [p = \mathsf{null}]$$
$$\mid X :: L' \;\Rightarrow\; \exists x p'.\;\; p \rightsquigarrow \{\!|\mathsf{hd}{=}x;\ \mathsf{tl}{=}p'|\!\} \star p' \rightsquigarrow \mathsf{MlistofMlist}\, L' \star x \rightsquigarrow \mathsf{MList}\, X$$

Generalization:

$$p \rightsquigarrow \mathsf{Mlistof}\, R\, L \;\equiv\; \mathsf{match}\, L\, \mathsf{with}$$
$$\mid \mathsf{nil} \;\Rightarrow\; [p = \mathsf{null}]$$
$$\mid X :: L' \;\Rightarrow\; \exists x p'.\;\; p \rightsquigarrow \{\!|\mathsf{hd}{=}x;\ \mathsf{tl}{=}p'|\!\} \star p' \rightsquigarrow \mathsf{Mlistof}\, R\, L' \star x \rightsquigarrow R\, X$$

In particular:

$$p \rightsquigarrow \mathsf{MlistofMlist}\, L \;=\; p \rightsquigarrow \mathsf{Mlistof}\, \mathsf{MList}\, L$$

Type-checking

$p \rightsquigarrow \mathsf{Mlistof}\, R\, L$	*is a notation for*	$\mathsf{Mlistof}\, R\, L\, p$	(of type Hprop)
$x \rightsquigarrow R\, X$	*is a notation for*	$R\, X\, x$	(of type Hprop)

$$
\begin{array}{rcl}
p \rightsquigarrow \mathsf{Mlistof}\, R\, L & \equiv & \mathsf{match}\ L\ \mathsf{with} \\
& & \mid \mathsf{nil} \Rightarrow [p = \mathsf{null}] \\
& & \mid X :: L' \Rightarrow \exists x p'.\quad p \rightsquigarrow \{\!|\mathsf{hd}{=}x;\ \mathsf{tl}{=}p'|\!\} \\
& & \qquad\qquad\qquad\quad \star\ p' \rightsquigarrow \mathsf{Mlistof}\, R\, L' \\
& & \qquad\qquad\qquad\quad \star\ x \rightsquigarrow R\, X
\end{array}
$$

Exercise: since $(p : \mathsf{loc})$ and $(x : \mathsf{Val})$ and $(X : A)$ for some A, what is the type of R? What is the type of Mlistof?

Type-checking

$p \rightsquigarrow \mathsf{Mlistof}\, R\, L$ *is a notation for* $\mathsf{Mlistof}\, R\, L\, p$ (of type Hprop)
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$$
\begin{array}{rcl}
p \rightsquigarrow \mathsf{Mlistof}\, R\, L & \equiv & \mathsf{match}\ L\ \mathsf{with} \\
& & \mid \mathsf{nil}\ \Rightarrow\ [p = \mathsf{null}] \\
& & \mid X :: L'\ \Rightarrow\ \exists x p'.\quad p \rightsquigarrow \{\!|\mathsf{hd}{=}x;\ \mathsf{tl}{=}p'|\!\} \\
& & \qquad\qquad\qquad\qquad \star\ p' \rightsquigarrow \mathsf{Mlistof}\, R\, L' \\
& & \qquad\qquad\qquad\qquad \star\ x \rightsquigarrow R\, X
\end{array}
$$

Exercise: since $(p : \mathsf{loc})$ and $(x : \mathsf{Val})$ and $(X : A)$ for some A, what is the type of R? What is the type of Mlistof?

- $R : A \to \mathsf{Val} \to \mathsf{Hprop}$
- $\mathsf{Mlistof} : \forall A.\ (A \to \mathsf{Val} \to \mathsf{Hprop}) \to \mathsf{list} A \to \mathsf{loc} \to \mathsf{Hprop}$

The identity representation predicate

$$\begin{aligned} p \leadsto \mathsf{Mlistof}\, R\, L \;\equiv\; & \mathsf{match}\, L\, \mathsf{with} \\ & \mid \mathsf{nil} \Rightarrow [p = \mathsf{null}] \\ & \mid X :: L' \Rightarrow \exists x p'.\quad p \leadsto \{\!|\mathsf{hd}{=}x;\ \mathsf{tl}{=}p'|\!\} \\ & \qquad\qquad\qquad\qquad \star\, p' \leadsto \mathsf{Mlistof}\, R\, L' \\ & \qquad\qquad\qquad\qquad \star\, x \leadsto R\, X \end{aligned}$$

$$\begin{aligned} p \leadsto \mathsf{MList}\, L \;\equiv\; & \mathsf{match}\, L\, \mathsf{with} \\ & \mid \mathsf{nil} \Rightarrow [p = \mathsf{null}] \\ & \mid x :: L' \Rightarrow \exists p'.\quad p \leadsto \{\!|\mathsf{hd}{=}x;\ \mathsf{tl}{=}p'|\!\} \\ & \qquad\qquad\qquad\quad \star\, p' \leadsto \mathsf{MList}\, L' \end{aligned}$$

Exercise: define the identity representation predicate Id such that

$$p \leadsto \mathsf{Mlistof}\,\mathsf{Id}\, L \;=\; p \leadsto \mathsf{MList}\, L$$

The identity representation predicate

$$
\begin{aligned}
p \leadsto \mathsf{Mlistof}\, R\, L \;\equiv\; & \mathsf{match}\, L\, \mathsf{with} \\
& \mid \mathsf{nil} \Rightarrow [p = \mathsf{null}] \\
& \mid X :: L' \Rightarrow \exists x p'.\quad p \leadsto \{\!|\mathsf{hd}{=}x;\ \mathsf{tl}{=}p'|\!\} \\
& \qquad\qquad\qquad\quad \star\ p' \leadsto \mathsf{Mlistof}\, R\, L' \\
& \qquad\qquad\qquad\quad \star\ x \leadsto R\, X
\end{aligned}
$$

$$
\begin{aligned}
p \leadsto \mathsf{MList}\, L \;\equiv\; & \mathsf{match}\, L\, \mathsf{with} \\
& \mid \mathsf{nil} \Rightarrow [p = \mathsf{null}] \\
& \mid x :: L' \Rightarrow \exists p'.\quad p \leadsto \{\!|\mathsf{hd}{=}x;\ \mathsf{tl}{=}p'|\!\} \\
& \qquad\qquad\qquad \star\ p' \leadsto \mathsf{MList}\, L'
\end{aligned}
$$

Exercise: define the identity representation predicate Id such that

$$p \leadsto \mathsf{Mlistof}\, \mathsf{Id}\, L \;=\; p \leadsto \mathsf{MList}\, L$$

Definition:

$$x \leadsto \mathsf{Id}\, X \;\equiv\; [x = X]$$

Summary

1. Higher-order predicate:

$$p \leadsto \mathsf{MList}\, L \qquad \text{is generalized into} \qquad p \leadsto \mathsf{Mlistof}\, R\, L$$

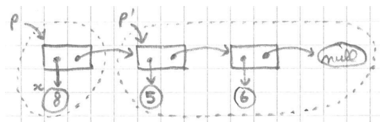
2. Identity representation predicate:

$$p \leadsto \mathsf{Mlistof}\, \mathsf{Id}\, L \qquad \text{is the same as} \qquad p \leadsto \mathsf{MList}\, L$$

Chapter 20

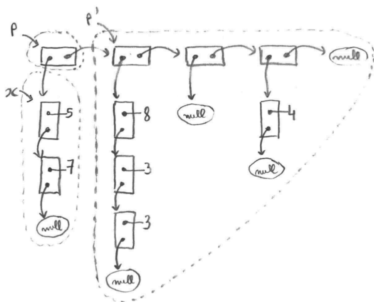
Higher-order representation predicates and the access problem

Specification of construction, for basic values



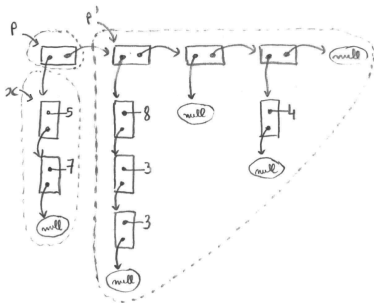
$$\{p' \rightsquigarrow \mathsf{MList}\, L\}\ (\mathsf{cons}\, x\, p')\ \{\lambda p.\ p \rightsquigarrow \mathsf{MList}\, (x :: L)\}$$

Specification of construction



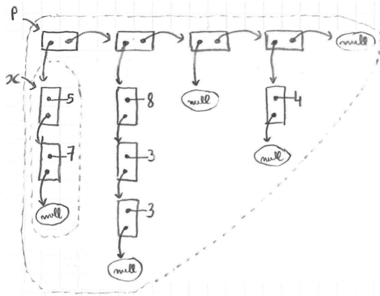
$$\{x \rightsquigarrow R\,X \;\star\; p' \rightsquigarrow \mathsf{Mlistof}\,R\,L\}\;(\mathsf{cons}\,x\,p')\;\{\lambda p.\; p \rightsquigarrow \mathsf{Mlistof}\,R\,(X :: L)\}$$

Specification of deconstruction



$$\{p \rightsquigarrow \mathsf{Mlistof}\, R\, (X :: L)\}\ (\mathsf{uncons}\, p)$$
$$\{\lambda(x, p').\ x \rightsquigarrow R\, X \star p' \rightsquigarrow \mathsf{Mlistof}\, R\, L\}$$

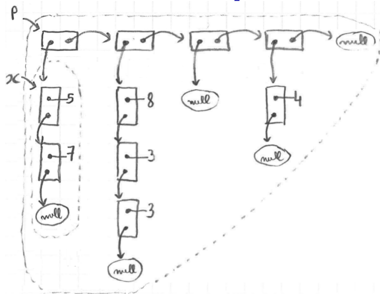
Specification of accesses: the problem



Incorrect specification for `head`:

$$\{p \leadsto \mathsf{Mlistof}\, R\, (X :: L)\}\ (\mathsf{head}\, p)$$
$$\{\lambda x.\ x \leadsto R\, X\ \star\ p \leadsto \mathsf{Mlistof}\, R\, (X :: L)\}$$

Specification of accesses: a partial solution



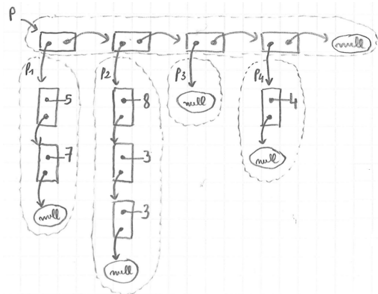
Correct yet limited specification:

$$\{p \leadsto \mathsf{Mlistof}\, R\, (X :: L)\}\ (\texttt{head}\, p)$$
$$\{\lambda x.\ x \leadsto R\, X \star (x \leadsto R\, X \mathrel{-\!\!\star} p \leadsto \mathsf{Mlistof}\, R\, (X :: L))\}$$

An intuition for $(\mathrel{-\!\!\star})$ is: $(H \star (H \mathrel{-\!\!\star} H')) \rhd H'$. More precisely:

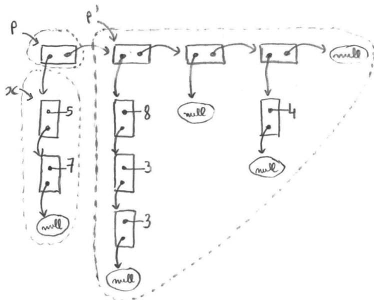
$$\frac{H_1 \star H_2 \vdash H_3}{H_1 \vdash (H_2 \mathrel{-\!\!\star} H_3)} \qquad \frac{H_1 \vdash (H_2 \mathrel{-\!\!\star} H_3) \qquad H_4 \vdash H_2}{H_1 \star H_4 \vdash H_3}$$

Specification of accesses: a brute force solution



$$
\begin{aligned}
p \rightsquigarrow \mathsf{Mlistof}\, R\, L \quad = \quad \exists K.\quad & p \rightsquigarrow \mathsf{MList}\, K \\
\star\ & \circledast_{i \,\in\, [0,\, |L|)}\ (K[i]) \rightsquigarrow R\,(L[i]) \\
\star\ & [\, |K| = |L| \,]
\end{aligned}
$$

Specification of accesses: focus before read



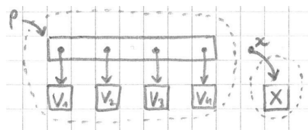
$$\begin{aligned} p \rightsquigarrow \mathsf{MIistof}\, R\, (X :: L) \;\;=\;\; \exists x p'.\quad & p \rightsquigarrow \{\!|\mathsf{hd}{=}x;\ \mathsf{tl}{=}p'|\!\} \\ \star\;\; & x \rightsquigarrow R\, X \\ \star\;\; & p' \rightsquigarrow \mathsf{MIistof}\, R\, L' \end{aligned}$$

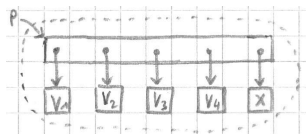
Then read using:

$$\{p \mapsto \{\!|\mathsf{hd}{=}x;\ \mathsf{tl}{=}p'|\!\}\}\ (p.\mathsf{hd})\ \{\lambda y.\ [y = x] \star p \mapsto \{\!|\mathsf{hd}{=}x;\ \mathsf{tl}{=}p'|\!\}\}$$

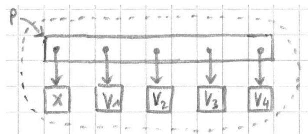
Ownership transfer with a queue of mutable items

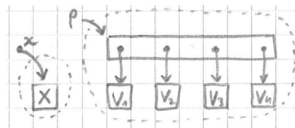
Push:





Pop:





Specification of queues of basic items

$\{[\,]\}\ (\texttt{create()})\ \{\lambda p.\ p \rightsquigarrow \mathsf{Queue}\,\mathsf{nil}\}$

$\{p \rightsquigarrow \mathsf{Queue}\,L\}\ (\texttt{push x p})\ \{\lambda_.\ p \rightsquigarrow \mathsf{Queue}\,(L\&x)\}$

$\{p \rightsquigarrow \mathsf{Queue}\,(x :: L)\}\ (\texttt{pop p})\ \{\lambda r.\ [r = x] \star p \rightsquigarrow \mathsf{Queue}\,L\}$

$\{p \rightsquigarrow \mathsf{Queue}\,L \star p' \rightsquigarrow \mathsf{Queue}\,L'\}\ (\texttt{concat p p'})\ \{\lambda_.\ p \rightsquigarrow \mathsf{Queue}\,(L +\!\!+ L')\}$

Specification of queues of mutable items

Exercise: specify functions over queues using a higher-order representation predicate written $p \rightsquigarrow \mathsf{Queueof}\, R\, L$.
Shorthand: just write "$\mathsf{Q}\, R$" instead of "$\mathsf{Queueof}\, R$".

Specification of queues of mutable items

Exercise: specify functions over queues using a higher-order representation predicate written $p \rightsquigarrow \mathsf{Queueof}\, R\, L$.
Shorthand: just write "$\mathsf{Q}\, R$" instead of "$\mathsf{Queueof}\, R$".

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$\{[\,]\}\ (\texttt{create()})\ \{\lambda p.\ p \leadsto \mathsf{Queueof}\, R\, \mathsf{nil}\}$

$\{p \leadsto \mathsf{Queueof}\, R\, L\ \star\ x \leadsto R\, X\}\ (\texttt{push}\, x\, p)\ \{\lambda_.\ p \leadsto \mathsf{Queueof}\, R\, (L \& X)\}$

Specification of queues of mutable items

Exercise: specify functions over queues using a higher-order representation predicate written $p \leadsto \mathsf{Queueof}\, R\, L$.
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$\{[\,]\}\ (\texttt{create()})\ \{\lambda p.\ p \leadsto \mathsf{Queueof}\, R\, \mathsf{nil}\}$

$\{p \leadsto \mathsf{Queueof}\, R\, L \star x \leadsto R\, X\}\ (\texttt{push}\, x\, p)\ \{\lambda _.\ p \leadsto \mathsf{Queueof}\, R\, (L \& X)\}$

$\{p \leadsto \mathsf{Queueof}\, R\, (X :: L)\}\ (\texttt{pop}\, p)\ \{\lambda x.\ p \leadsto \mathsf{Queueof}\, R\, L \star x \leadsto R\, X\}$

Specification of queues of mutable items

Exercise: specify functions over queues using a higher-order representation predicate written $p \leadsto \mathsf{Queueof}\, R\, L$.
Shorthand: just write "Q R" instead of "Queueof R".

$$\{[\,]\}\ (\texttt{create()})\ \{\lambda p.\ p \leadsto \mathsf{Queueof}\, R\, \mathsf{nil}\}$$

$$\{p \leadsto \mathsf{Queueof}\, R\, L \star x \leadsto R\, X\}\ (\texttt{push}\, x\, p)\ \{\lambda_.\ p \leadsto \mathsf{Queueof}\, R\, (L \& X)\}$$

$$\{p \leadsto \mathsf{Queueof}\, R\, (X :: L)\}\ (\texttt{pop}\, p)\ \{\lambda x.\ p \leadsto \mathsf{Queueof}\, R\, L \star x \leadsto R\, X\}$$

$$\{p \leadsto \mathsf{Queueof}\, R\, L \star p' \leadsto \mathsf{Queueof}\, R\, L'\}\ (\texttt{concat}\, p\, p')\ \{\lambda_.\ p \leadsto \mathsf{Queueof}\, R\, (L +\!\!+ L')\}$$

The copy problem

Incorrect specification for copy:

$$\{p \leadsto \mathsf{Queueof}\, R\, L\}$$
$$(\mathsf{copy}\, p)$$
$$\{\lambda p'.\ p \leadsto \mathsf{Queueof}\, R\, L \star p' \leadsto \mathsf{Queueof}\, R\, L\}$$

Exercise: specify a function $\mathtt{copy}\, f\, p$ that duplicables a mutable queue specified using Queueof, where f is a function to duplicate items.

The copy problem

Incorrect specification for `copy`:

$$\{p \rightsquigarrow \mathsf{Queueof}\, R\, L\}$$
$$(\mathsf{copy}\, p)$$
$$\{\lambda p'.\ p \rightsquigarrow \mathsf{Queueof}\, R\, L \star p' \rightsquigarrow \mathsf{Queueof}\, R\, L\}$$

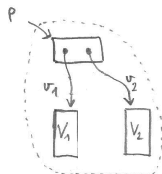
Exercise: specify a function $\texttt{copy}\, f\, p$ that duplicables a mutable queue specified using Queueof, where f is a function to duplicate items.

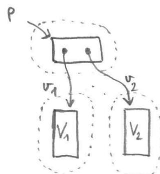
$$\big(\forall x X.\ \{x \rightsquigarrow R\, X\}\ (f\, x)\ \{\lambda x'.\ x \rightsquigarrow R\, X \star x' \rightsquigarrow R\, X\}\big)$$
$$\Rightarrow\ \{p \rightsquigarrow \mathsf{Queueof}\, R\, L\}$$
$$(\mathsf{copy}\, f\, p)$$
$$\{\lambda p'.\ p \rightsquigarrow \mathsf{Queueof}\, R\, L \star p' \rightsquigarrow \mathsf{Queueof}\, R\, L\}$$

Chapter 21

Higher-order representation predicates for records

Representation for records





$$
\begin{aligned}
p \leadsto \mathsf{MCellof}\, R_1\, V_1\, R_2\, V_2 \;\equiv\; \exists v_1 v_2.\quad & p \leadsto \{\!|\mathsf{hd}{=}v_1;\ \mathsf{tl}{=}v_2|\!\} \\
\star\ & v_1 \leadsto R_1\, V_1 \\
\star\ & v_2 \leadsto R_2\, V_2
\end{aligned}
$$

Representation predicate for lists, revisited

$$
\begin{aligned}
p \rightsquigarrow \mathsf{Mlistof}\, R\, L \;\equiv\;& \mathsf{match}\, L\, \mathsf{with} \\
& \mid \mathsf{nil} \Rightarrow [p = \mathsf{null}] \\
& \mid X :: L' \Rightarrow \exists x p'.\quad p \rightsquigarrow \{\!|\mathsf{hd}{=}x;\, \mathsf{tl}{=}p'|\!\} \\
& \qquad\qquad\qquad\quad \star\, x \rightsquigarrow R\, X \\
& \qquad\qquad\qquad\quad \star\, p' \rightsquigarrow \mathsf{Mlistof}\, R\, L'
\end{aligned}
$$

$$
\begin{aligned}
p \rightsquigarrow \mathsf{MCellof}\, R_1\, V_1\, R_2\, V_2 \;\equiv\; \exists v_1 v_2.\quad & p \rightsquigarrow \{\!|\mathsf{hd}{=}v_1;\, \mathsf{tl}{=}v_2|\!\} \\
& \star\, v_1 \rightsquigarrow R_1\, V_1 \\
& \star\, v_2 \rightsquigarrow R_2\, V_2
\end{aligned}
$$

Exercise: rewrite the specification of Mlistof using MCellof.

Representation predicate for lists, revisited

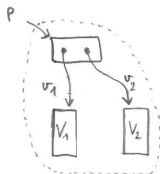
$$
\begin{array}{rcl}
p \leadsto \mathsf{MlistOf}\, R\, L & \equiv & \mathsf{match}\, L\, \mathsf{with} \\
 & & \mid \mathsf{nil} \;\Rightarrow\; [p = \mathsf{null}] \\
 & & \mid X :: L' \;\Rightarrow\; \exists x p'.\quad p \leadsto \{\!|\mathsf{hd}{=}x;\ \mathsf{tl}{=}p'|\!\} \\
 & & \qquad\qquad\qquad\qquad \star\ x \leadsto R\, X \\
 & & \qquad\qquad\qquad\qquad \star\ p' \leadsto \mathsf{MlistOf}\, R\, L'
\end{array}
$$

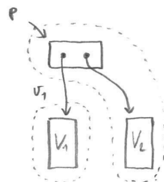
$$
\begin{array}{rcl}
p \leadsto \mathsf{MCellOf}\, R_1\, V_1\, R_2\, V_2 & \equiv & \exists v_1 v_2.\quad p \leadsto \{\!|\mathsf{hd}{=}v_1;\ \mathsf{tl}{=}v_2|\!\} \\
 & & \qquad\quad \star\ v_1 \leadsto R_1\, V_1 \\
 & & \qquad\quad \star\ v_2 \leadsto R_2\, V_2
\end{array}
$$

Exercise: rewrite the specification of MlistOf using MCellOf.

$$
\begin{array}{rcl}
p \leadsto \mathsf{MlistOf}\, R\, L & \equiv & \mathsf{match}\, L\, \mathsf{with} \\
 & & \mid \mathsf{nil} \;\Rightarrow\; [p = \mathsf{null}] \\
 & & \mid X :: L' \;\Rightarrow\; p \leadsto \mathsf{MCellOf}\, R\, X\, (\mathsf{MlistOf}\, R)\, L'
\end{array}
$$

Focus/unfocus for accessing a record field

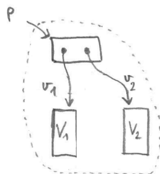


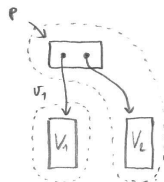


Focus on a field:

$$p \rightsquigarrow \mathsf{MCellof}\, R_1\, V_1\, R_2\, V_2 \;=\; \exists v_1.\; p \rightsquigarrow \mathsf{MCellof}\, \mathsf{Id}\, v_1\, R_2\, V_2 \;\star\; v_1 \rightsquigarrow R_1\, V_1$$

Focus/unfocus for accessing a record field





Focus on a field:

$$p \rightsquigarrow \mathsf{MCellof}\, R_1\, V_1\, R_2\, V_2 \;\; = \;\; \exists v_1.\; p \rightsquigarrow \mathsf{MCellof}\,\mathsf{Id}\, v_1\, R_2\, V_2 \;\star\; v_1 \rightsquigarrow R_1\, V_1$$

Access to a focused field:

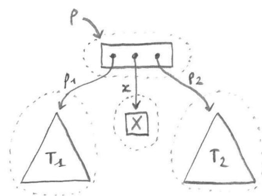
$$\{p \rightsquigarrow \mathsf{MCellof}\,\mathsf{Id}\, v_1\, R_2\, V_2\}\; (p.\mathsf{hd})\; \{\lambda x.\, [x = v_1] \star p \rightsquigarrow \mathsf{MCellof}\,\mathsf{Id}\, v_1\, R_2\, V_2\}$$

$$\{p \rightsquigarrow \mathsf{MCellof}\,\mathsf{Id}\, v_1\, R_2\, V_2\}\; (p.\mathsf{hd} \texttt{<-}\, w)\; \{\lambda_.\; p \rightsquigarrow \mathsf{MCellof}\,\mathsf{Id}\, w\, R_2\, V_2\}$$

Chapter 22

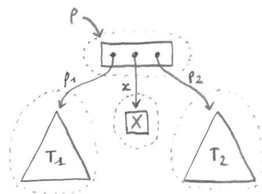
Higher-order representation predicates for trees

Binary tree: representation



$$
\begin{aligned}
p \leadsto \mathsf{Mtreeof}\, R\, T \;\equiv\; & \mathsf{match}\, T\, \mathsf{with} \\
& \mid \mathsf{Leaf} \;\Rightarrow\; [p = \mathsf{null}] \\
& \mid \mathsf{Node}\, X\, T_1\, T_2 \;\Rightarrow\; \exists x p_1 p_2. \\
& \qquad\quad p \mapsto \{\!|\mathsf{item}{=}x;\ \mathsf{left}{=}p_1;\ \mathsf{right}{=}p_2|\!\} \\
& \qquad \star\ x \leadsto R\, X \\
& \qquad \star\ p_1 \leadsto \mathsf{Mtreeof}\, R\, T_1 \\
& \qquad \star\ p_2 \leadsto \mathsf{Mtreeof}\, R\, T_2
\end{aligned}
$$

Binary tree: representation, revisited

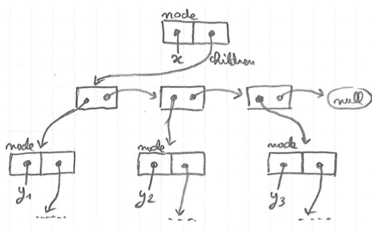


Representation predicate for tree cells:

$$p \rightsquigarrow \mathsf{Nodeof}\, R_1\, V_1\, R_2\, V_2\, R_3\, V_3 \quad \equiv$$
$$\exists v_1 v_2 v_3. \quad p \mapsto \{\!|\mathsf{item}{=}v_1;\ \mathsf{left}{=}v_2;\ \mathsf{right}{=}v_3|\!\}$$
$$\star\ v_1 \rightsquigarrow R_1\, V_1\ \star\ v_2 \rightsquigarrow R_2\, V_2\ \star\ v_3 \rightsquigarrow R_3\, V_3$$

$$p \rightsquigarrow \mathsf{Mtreeof}\, R\, T \ \equiv\ \mathsf{match}\, T\, \mathsf{with}$$
$$|\ \mathsf{Leaf}\ \Rightarrow\ [p = \mathsf{null}]$$
$$|\ \mathsf{Node}\, X\, T_1\, T_2\ \Rightarrow$$
$$p \rightsquigarrow \mathsf{Nodeof}\, R\, X\, (\mathsf{Mtreeof}\, R)\, T_1\, (\mathsf{Mtreeof}\, R)\, T_2$$

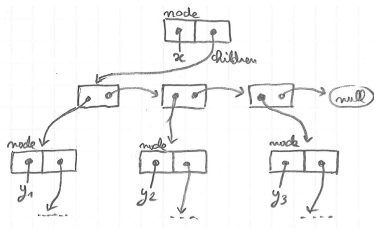
Trees with list of subtrees: implementation




```
type 'a node = {
  mutable item : 'a;
  mutable children : ('a node) cell }
```

```
Inductive tree (A:Type) : Type :=
  | Leaf : tree A
  | Node : A → list (tree A) → tree A.
```

Trees with list of subtrees: specification



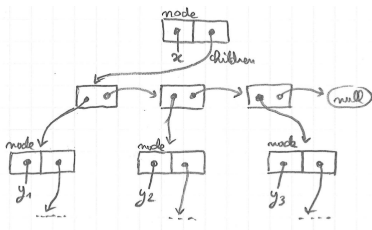
$$
\begin{array}{l}
p \rightsquigarrow \mathsf{Narytreeof}\, R\, T \equiv \\
\quad \mathsf{match}\, T\, \mathsf{with} \\
\quad \mid \mathsf{Leaf} \Rightarrow [p = \mathsf{null}] \\
\quad \mid \mathsf{Node}\, X\, L \Rightarrow \exists xc. \quad p \mapsto \{\!|\mathsf{item}{=}x;\ \mathsf{children}{=}c|\!\} \\
\qquad\qquad\qquad\qquad\quad \star\ x \rightsquigarrow R\, X \\
\qquad\qquad\qquad\qquad\quad \star\ c \rightsquigarrow \mathsf{Mlistof}\,(\mathsf{Narytreeof}\, R)\, L
\end{array}
$$

Trees with list of subtrees: representation of nodes

$$
\begin{aligned}
p \leadsto \mathsf{Nodeof}\, R_1\, V_1\, R_2\, V_2 \quad \equiv \\
\exists v_1 v_2. \quad & p \mapsto \{\!|\mathsf{item}{=}v_1;\ \mathsf{children}{=}v_2|\!\} \\
& \star\, v_1 \leadsto R_1\, V_1 \\
& \star\, v_2 \leadsto R_2\, V_2
\end{aligned}
$$

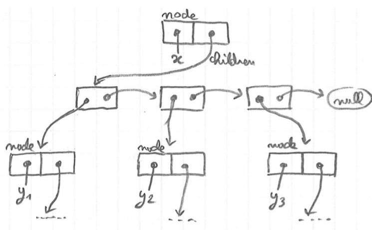
$$
\begin{aligned}
& p \leadsto \mathsf{Narytreeof}\, R\, T \equiv \\
& \quad \mathsf{match}\, T\, \mathsf{with} \\
& \quad \mid \mathsf{Leaf} \;\Rightarrow\; [p = \mathsf{null}] \\
& \quad \mid \mathsf{Node}\, X\, L \;\Rightarrow\; \exists xc. \quad p \mapsto \{\!|\mathsf{item}{=}x;\ \mathsf{children}{=}c|\!\} \\
& \qquad\qquad\qquad\qquad\quad \star\, x \leadsto R\, X \\
& \qquad\qquad\qquad\qquad\quad \star\, c \leadsto \mathsf{Mlistof}\, (\mathsf{Narytreeof}\, R)\, L
\end{aligned}
$$

Trees with list of subtrees, revisited



Exercise: rewrite the specification of Narytreeof using Nodeof.

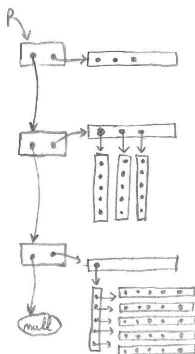
Trees with list of subtrees, revisited



Exercise: rewrite the specification of Narytreeof using Nodeof.

$$
\begin{aligned}
& p \leadsto \mathsf{Narytreeof}\, R\, T \equiv \\
& \quad \mathsf{match}\, T\, \mathsf{with} \\
& \quad \mid \mathsf{Leaf} \;\Rightarrow\; [p = \mathsf{null}] \\
& \quad \mid \mathsf{Node}\, X\, L \;\Rightarrow\; p \leadsto \mathsf{Nodeof}\, R\, X\, (\mathsf{Mlistof}\, (\mathsf{Narytreeof}\, R))\, L
\end{aligned}
$$

More involved application



- Exam from 2015, Exercise 3: bootstrapped chunked bags.

Available from the webpage of the course.

Chapter 23

Iteration with higher-order representation predicates

Iteration on lists

Recall:

$$\forall f l I.\quad \big(\forall x k.\ \{I\,k\}\ (f\,x)\ \{\lambda_.\ I\,(k\&x)\}\big) \\ \Rightarrow\quad \{I\,\mathsf{nil}\}\ (\mathtt{iter}\,f\,l)\ \{\lambda_.\ I\,l\}$$

$$\forall f p l I.\quad \big(\forall x k.\ \{I\,k\}\ (f\,x)\ \{\lambda_.\ I\,(k\&x)\}\big) \\ \Rightarrow\quad \{p \leadsto \mathsf{MList}\,l\ \star\ I\,\mathsf{nil}\}\ (\mathtt{miter}\,f\,p)\ \{\lambda_.\ p \leadsto \mathsf{MList}\,l\ \star\ I\,l\}$$

Iteration on lists

Recall:

$$\forall f l I. \quad \big(\forall x k.\ \{I\,k\}\ (f\,x)\ \{\lambda_.\ I\,(k\&x)\}\big) \\ \Rightarrow \quad \{I\,\mathsf{nil}\}\ (\texttt{iter}\,f\,l)\ \{\lambda_.\ I\,l\}$$

$$\forall f p l I. \quad \big(\forall x k.\ \{I\,k\}\ (f\,x)\ \{\lambda_.\ I\,(k\&x)\}\big) \\ \Rightarrow \quad \{p \rightsquigarrow \mathsf{MList}\,l\ \star\ I\,\mathsf{nil}\}\ (\texttt{miter}\,f\,p)\ \{\lambda_.\ p \rightsquigarrow \mathsf{MList}\,l\ \star\ I\,l\}$$

Challenge:

$$\big(\forall x....\ \{...\}\ (f\,x)\ \{\lambda_.\ ...\}\big) \\ \Rightarrow \quad \{p \rightsquigarrow \mathsf{Mlistof}\,R\,L\ \star\ ...\}\ (\texttt{miter}\,f\,p)\ \{\lambda_.\ p \rightsquigarrow\ ...\ \star\ ...\}$$

Question: Can we use an invariant $I \equiv \lambda K.(\dots)$?
(i.e. with a spec of the form $\{p \rightsquigarrow \dots \star I\ \mathsf{nil}\}(\dots)\{p \rightsquigarrow \dots \star I\ L\}$?)

Iterating over a mutable list of mutable items

Exercise: specify the function `miter`, using an invariant of the form $J\,K\,K'$, describing the state before and the state after the iteration.

Iterating over a mutable list of mutable items

Exercise: specify the function `miter`, using an invariant of the form $J\,K\,K'$, describing the state before and the state after the iteration.

$$\begin{array}{rl} \forall fpRLJ. & \Big(\forall xXKK'.\ \{x \rightsquigarrow R\,X \star J\,K\,K'\} \\ & \qquad\qquad (f\,x) \\ & \qquad\qquad \{\lambda_.\ \exists X'.\ x \rightsquigarrow R\,X' \star J\,(K\&X)\,(K'\&X')\}\Big) \\ \Rightarrow & \{p \rightsquigarrow \mathsf{Mlistof}\,R\,L \star J\,\mathsf{nil}\,\mathsf{nil}\} \\ & (\mathtt{miter}\,f\,p) \\ & \{\lambda_.\ \exists L'.\ p \rightsquigarrow \mathsf{Mlistof}\,R\,L' \star J\,L\,L'\} \end{array}$$

Incrementing a mutable list of distinct references (1/2)

```
let incr_all p =
  miter (fun x -> incr x) p

let example_p =
  { hd = ref 5; tl = { hd = ref 3; tl = null } }
```

$$x \rightsquigarrow \mathsf{Ref}\, X \;\equiv\; x \mapsto X$$

Exercise: using the representation predicates Ref and Mlistof, specify the function `(fun x -> incr x)` and `incr_all`.

Incrementing a mutable list of distinct references (1/2)

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Exercise: using the representation predicates Ref and Mlistof, specify the function `(fun x -> incr x)` and `incr_all`.

$$\{x \leadsto \mathsf{Ref}\, X\}\; (\texttt{incr x})\; \{\lambda_.\; x \leadsto \mathsf{Ref}\,(X+1)\}$$

Incrementing a mutable list of distinct references (1/2)

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Exercise: using the representation predicates Ref and Mlistof, specify the function `(fun x -> incr x)` and `incr_all`.

$$\{x \rightsquigarrow \mathsf{Ref}\, X\}\ (\texttt{incr x})\ \{\lambda_.\ x \rightsquigarrow \mathsf{Ref}\,(X+1)\}$$

$$\{p \rightsquigarrow \mathsf{Mlistof}\,\mathsf{Ref}\,L\}\ (\texttt{incr_all}\, p)\ \{\lambda_.\ p \rightsquigarrow \mathsf{Mlistof}\,\mathsf{Ref}\,(\mathsf{map}\,(+1)\,L)\}$$

Incrementing a mutable list of distinct references (2/2)

$$\begin{array}{ll} \forall f p R L J. & \big(\forall x X K K'.\ \{x \rightsquigarrow R\, X \star J\, K\, K'\} \\ & \qquad\qquad (f\, x) \\ & \qquad\qquad \{\lambda_.\ \exists X'.\ x \rightsquigarrow R\, X' \star J\,(K \& X)\,(K' \& X')\}\big) \\ \Rightarrow & \{p \rightsquigarrow \mathsf{Mlistof}\, R\, L \star J\, \mathsf{nil}\, \mathsf{nil}\} \\ & (\mathtt{miter}\, f\, p) \\ & \{\lambda_.\ \exists L'.\ p \rightsquigarrow \mathsf{Mlistof}\, R\, L' \star J\, L\, L'\} \end{array}$$

Consider:

$$J\, K\, K' \;\equiv\; [K' = \mathsf{map}\,(+1)\, K]$$

Derives:

$$\big(\forall x X.\ \{x \rightsquigarrow \mathsf{Ref}\, X\}\ (\mathtt{fun\ x\ ->\ incr\ x})x\ \{\lambda_.\ x \rightsquigarrow \mathsf{Ref}\,(X+1)\}\big) \Rightarrow$$

$$\{p \rightsquigarrow \mathsf{Mlistof}\, \mathsf{Ref}\, L\}\ (\mathtt{incr_all}\, p)\ \{\lambda_.\ p \rightsquigarrow \mathsf{Mlistof}\, \mathsf{Ref}\,(\mathsf{map}\,(+1)\, L)\}$$

Chapter 24

Resource analysis in Separation Logic

Controlling deallocation

(1) Remove the garbage collection rule:

$$\frac{\{H\}\ t\ \{Q \star \mathsf{GC}\}}{\{H\}\ t\ \{Q\}}\ \textsc{gc-post}$$

(2) Add a "free" function for explicit deallocation:

$$\{r \mapsto v\}\ (\texttt{free r})\ \{\lambda_.\ [\,]\}$$

Controlling deallocation

(1) Remove the garbage collection rule:

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(3) Theorem: for a full program execution starting in the empty heap, all the data still allocated at the end is described in the post-condition.

(4) Corollary: terminating on the empty heap ensures no memory leaks.

$$\{[\,]\}\ t\ \{\lambda n.\ [P\,n]\}$$

Controlling deallocation

(1) Remove the garbage collection rule:

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(3) Theorem: for a full program execution starting in the empty heap, all the data still allocated at the end is described in the post-condition.

(4) Corollary: terminating on the empty heap ensures no memory leaks.

$$\{[\,]\}\ t\ \{\lambda n.\ [P\, n]\}$$

(note: a separation logic with GC can be called “affine” or “intuitionistic”)

File handle protocols

Goal: ensure that if a file is open then it is eventually closed.

$$f \rightsquigarrow \mathsf{File}\, L$$

where $(f : \mathsf{loc})$ denotes the file handler,
and $(L : \mathsf{list}\,\mathsf{char})$ denotes the remaining bytes to read.

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$$\{[\,]\}\;\; (\texttt{fopen s})\;\; \{\lambda f.\; \exists L.\; f \leadsto \mathsf{File}\, L\}$$

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$$\{[\,]\}\ \ (\texttt{fopen s})\ \ \{\lambda f.\ \exists L.\ f \leadsto \mathsf{File}\, L\}$$
$$\{f \leadsto \mathsf{File}\,(c :: L)\}\ \ (\texttt{fread f})\ \ \{\lambda x.\ [x = c] \star f \leadsto \mathsf{File}\, L\}$$

File handle protocols

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$$f \rightsquigarrow \mathsf{File}\, L$$

where $(f : \mathsf{loc})$ denotes the file handler,
and $(L : \mathsf{list}\,\mathsf{char})$ denotes the remaining bytes to read.

$$\begin{aligned}
\{[\,]\}\ \ (\texttt{fopen s})\ \ \{\lambda f.\ \exists L.\ f \rightsquigarrow \mathsf{File}\, L\} \\
\{f \rightsquigarrow \mathsf{File}\,(c :: L)\}\ \ (\texttt{fread f})\ \ \{\lambda x.\ [x = c] \star f \rightsquigarrow \mathsf{File}\, L\} \\
\{f \rightsquigarrow \mathsf{File}\, L\}\ (\texttt{fclose f})\ \{\lambda_.\ [\,]\}
\end{aligned}$$

Complexity analysis

Time credits:

$$\$\,x \quad : \quad \text{Hprop} \qquad\qquad \text{where } x \in \mathbb{R}^+$$

Properties:

$$\$(x+y) \;=\; \$\,x \;\star\; \$\,y \qquad \text{and} \qquad \$\,0 \;=\; [\,]$$

Complexity analysis

Time credits:

$$\$\,x \ : \ \text{Hprop} \qquad \text{where } x \in \mathbb{R}^+$$

Properties:

$$\$(x+y) = \$\,x \ \star \ \$\,y \quad \text{and} \quad \$\,0 = [\,]$$

Principle:

The execution of every instruction costs $\$1$.

Simplification:

Entering the body of a function or a loop costs $\$1$.

Time credits in pre-conditions

Constant time:

$$\{t \rightsquigarrow \mathsf{Array}\, M \star \$\, c\}\ (\texttt{Array.length t})\ \{\lambda n.\, [n = |M|] \star t \rightsquigarrow \mathsf{Array}\, M\}$$

Time credits in pre-conditions

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$$\{t \rightsquigarrow \mathsf{Array}\, M \star \$\, c\}\ (\texttt{Array.length t})\ \{\lambda n.\, [n = |M|] \star t \rightsquigarrow \mathsf{Array}\, M\}$$

Linear time:

$$\{\$(c_1 n + c_2)\}\ (\texttt{Array.make n v})\ \{\lambda t.\ \exists L.\ t \rightsquigarrow \mathsf{Array}\, L \star [...]\}$$

Time credits in pre-conditions

Constant time:

$$\{t \rightsquigarrow \mathsf{Array}\, M \star \$\, c\}\ (\texttt{Array.length t})\ \{\lambda n.\, [n = |M|] \star t \rightsquigarrow \mathsf{Array}\, M\}$$

Linear time:

$$\{\$(c_1 n + c_2)\}\ (\texttt{Array.make n v})\ \{\lambda t.\ \exists L.\ t \rightsquigarrow \mathsf{Array}\, L \star [\ldots]\}$$

Quasilinear time:

$$\{t \rightsquigarrow \mathsf{Array}\, L \star \$(c_1 |L| \log |L| + c_2)\}$$
$$(\texttt{Array.sort t})$$
$$\{\lambda t.\ \exists L'.\ t \rightsquigarrow \mathsf{Array}\, L' \star [\ldots]\}$$

Amortized analysis

Stack of unbounded size with amortized constant-time operations:

$\{\$\, c\}$ (`Stack.create()`) $\{\lambda_.\, s \rightsquigarrow \mathsf{Stack}\,\mathsf{nil}\}$

$\{s \rightsquigarrow \mathsf{Stack}\, L \star \$\, c\}$ (`Stack.push s x`) $\{\lambda_.\, s \rightsquigarrow \mathsf{Stack}\,(x :: L)\}$

$\{s \rightsquigarrow \mathsf{Stack}\,(x :: L) \star \$\, c\}$ (`Stack.pop s`) $\{\lambda y.\ [y = x] \star s \rightsquigarrow \mathsf{Stack}\, L\}$

Amortized analysis

Stack of unbounded size with amortized constant-time operations:

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$\{s \rightsquigarrow \mathsf{Stack}\, L \star \$\, c\}$ (`Stack.push s x`) $\{\lambda_.\, s \rightsquigarrow \mathsf{Stack}\,(x :: L)\}$

$\{s \rightsquigarrow \mathsf{Stack}\,(x :: L) \star \$\, c\}$ (`Stack.pop s`) $\{\lambda y.\ [y = x] \star s \rightsquigarrow \mathsf{Stack}\, L\}$

Representation predicate with a potential function:

$$
\begin{aligned}
s \rightsquigarrow \mathsf{Stack}\, L \quad \equiv \quad \exists ntMk. \quad & s \mapsto \{\!|\mathsf{size}{=}n;\ \mathsf{data}{=}t|\!\} \\
& \star\ t \rightsquigarrow \mathsf{Array}\, M \\
& \star\ [n = |L| \leqslant |M| = 2^k] \\
& \star\ [\forall i \in [0, n).\ M[i] = L[i]] \\
& \star\ \$(c' \cdot \mathsf{abs}(n - |M|/2))
\end{aligned}
$$

Chapter 25

Read-only permissions

Motivation for read-only permissions

What we currently need to write:

$$\{a_1 \leadsto \mathsf{Array}\, L_1 \star a_2 \leadsto \mathsf{Array}\, L_2\}$$
$$(\texttt{concat}\, a_1\, a_2)$$
$$\{\lambda a_3.\ a_3 \leadsto \mathsf{Array}\,(L_1 +\!\!+ L_2) \star a_1 \leadsto \mathsf{Array}\, L_1 \star a_2 \leadsto \mathsf{Array}\, L_2\}$$

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What we wish to write:

$$\{a_1 \overset{\mathsf{ro}}{\leadsto} \mathsf{Array}\, L_1 \star a_2 \overset{\mathsf{ro}}{\leadsto} \mathsf{Array}\, L_2\}$$
$$(\mathtt{concat}\, a_1\, a_2)$$
$$\{\lambda a_3.\ a_3 \leadsto \mathsf{Array}\, (L_1 +\!\!+ L_2)\}$$

Motivation for read-only permissions

What we currently need to write:

$$\{a_1 \rightsquigarrow \mathsf{Array}\, L_1 \star a_2 \rightsquigarrow \mathsf{Array}\, L_2\}$$
$$(\mathtt{concat}\, a_1\, a_2)$$
$$\{\lambda a_3.\ a_3 \rightsquigarrow \mathsf{Array}\,(L_1 +\!\!+ L_2) \star a_1 \rightsquigarrow \mathsf{Array}\, L_1 \star a_2 \rightsquigarrow \mathsf{Array}\, L_2\}$$

What we wish to write:

$$\{a_1 \overset{\mathsf{ro}}{\rightsquigarrow} \mathsf{Array}\, L_1 \star a_2 \overset{\mathsf{ro}}{\rightsquigarrow} \mathsf{Array}\, L_2\}$$
$$(\mathtt{concat}\, a_1\, a_2)$$
$$\{\lambda a_3.\ a_3 \rightsquigarrow \mathsf{Array}\,(L_1 +\!\!+ L_2)\}$$

More than syntactic sugar:
– we wish “ro” to enforce no write operations,
– we wish to allow aliasing of read-only arguments.

Fractional permissions

$$(r \overset{\alpha}{\mapsto} v) \qquad \text{with } \; 0 < \alpha \leqslant 1$$

Splitting and merging:

$$(r \mapsto v) \;=\; (r \overset{1}{\mapsto} v) \;=\; (r \overset{1/2}{\mapsto} v) \star (r \overset{1/2}{\mapsto} v)$$

More generally:

$$(r \overset{\alpha+\beta}{\mapsto} v) \;=\; (r \overset{\alpha}{\mapsto} v) \star (r \overset{\beta}{\mapsto} v) \qquad \text{with } \; 0 < \alpha, \beta \leqslant 1$$

Fractional permissions

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Splitting and merging:

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More generally:

$$(r \overset{\alpha+\beta}{\mapsto} v) \ = \ (r \overset{\alpha}{\mapsto} v) \star (r \overset{\beta}{\mapsto} v) \qquad \text{with } \ 0 < \alpha, \beta \leqslant 1$$

Operations:

$$\{[\,]\} \ (\texttt{ref v}) \ \{\lambda r.\ r \overset{1}{\mapsto} v\}$$

$$\{r \overset{1}{\mapsto} v'\} \ (\texttt{r := v}) \ \{\lambda_.\ r \overset{1}{\mapsto} v\}$$

$$\forall \alpha. \qquad \{r \overset{\alpha}{\mapsto} v\} \quad (\texttt{!r}) \quad \{\lambda x.\ [x = v] \star (r \overset{\alpha}{\mapsto} v)\}$$

Fractional permissions in practice

$$\forall \alpha\beta.\ \{a_1 \overset{\alpha}{\rightsquigarrow} \mathsf{Array}\, L_1 \star a_2 \overset{\beta}{\rightsquigarrow} \mathsf{Array}\, L_2\}$$
$$(\texttt{concat}\, a_1\, a_2)$$
$$\{\lambda a_3.\ a_1 \overset{\alpha}{\rightsquigarrow} \mathsf{Array}\, L_1 \star a_2 \overset{\beta}{\rightsquigarrow} \mathsf{Array}\, L_2 \star a_3 \overset{1}{\rightsquigarrow} \mathsf{Array}\,(L_1 +\!\!+ L_2)\}$$

Fractional permissions in practice

$$\forall \alpha\beta.\ \{a_1 \overset{\alpha}{\rightsquigarrow} \mathsf{Array}\, L_1 \star a_2 \overset{\beta}{\rightsquigarrow} \mathsf{Array}\, L_2\}$$
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$$\{\lambda a_3.\ a_1 \overset{\alpha}{\rightsquigarrow} \mathsf{Array}\, L_1 \star a_2 \overset{\beta}{\rightsquigarrow} \mathsf{Array}\, L_2 \star a_3 \overset{1}{\rightsquigarrow} \mathsf{Array}\, (L_1 +\!\!+ L_2)\}$$

Limitations:

– need to quantify fractions explicitly,

– need to syntactic sugar to avoid copy-pasting,

– need to re-establish post-conditions,

– a fraction $\frac{1}{2}H$ cannot be defined for arbitrary H.

Generic read-only modifier

Extension of the logic with a modifier $\mathsf{RO}(H)$ that applies to any H.

$$a \overset{\mathsf{ro}}{\rightsquigarrow} \mathsf{Array}\, L \;\equiv\; \mathsf{RO}(a \rightsquigarrow \mathsf{Array}\, L)$$

$\mathsf{RO}(H)$ is duplicatable and never mentioned in post-conditions.

$$\frac{}{\mathsf{RO}(H) \;\;\triangleright\;\; \mathsf{RO}(H) \,\star\, \mathsf{RO}(H)}\ \textsc{dup-ro}$$

$$\frac{}{\{\mathsf{RO}(l \mapsto v)\}\ (\mathsf{get}\, l)\ \{\lambda x.\ [x = v]\}}\ \textsc{get-ro}$$

Read-only frame rule

$\mathsf{RO}(H)$ is introduced on frame:

$$\frac{\{H \star \mathsf{RO}(H')\}\ t\ \{Q\} \qquad \text{no-ro-in}\ H'}{\{H \star H'\}\ t\ \{Q \star H'\}}\ \textsc{frame-ro}$$

Read-only sequencing rule

$$\frac{\{H\}\ t_1\ \{Q'\} \qquad \{Q'()\}\ t_2\ \{Q\}}{\{H\}\ (t_1\ ;\ t_2)\ \{Q\}}\ \textsc{seq}$$

$$\frac{\{H \star \mathsf{RO}(H')\}\ t_1\ \{Q'\} \qquad \{Q'() \star \mathsf{RO}(H')\}\ t_2\ \{Q\}}{\{H \star \mathsf{RO}(H')\}\ (t_1\ ;\ t_2)\ \{Q\}}\ \textsc{seq-ro}$$

$$\frac{\{H\}\ t_1\ \{Q'\} \qquad \{Q'() \star H'\}\ t_2\ \{Q\}}{\{H \star H'\}\ (t_1\ ;\ t_2)\ \{Q\}}\ \textsc{seq-frame}$$

RO in practice

$$\{\mathsf{RO}(a_1 \rightsquigarrow \mathsf{Array}\, L_1) \star \mathsf{RO}(a_2 \rightsquigarrow \mathsf{Array}\, L_2)\}$$
$$(\texttt{concat}\, a_1\, a_2)$$
$$\{\lambda a_3.\ a_3 \rightsquigarrow \mathsf{Array}\,(L_1 \mathbin{+\!\!+} L_2)\}$$

Chapter 26

Parallelism and Concurrency

Parallel pairs

A parallel pair, written $(|t_1, t_2|)$, for evaluating two subterms in parallel.
(Note: one often sees $t_1||t_2$ for let $((),()) = (|t_1, t_2|)$ in $()$.)

Computing: $a[i] + a[i+1] + ... + a[j-1]$.

```
let rec sum a i j =
  if j - i = 1 then a.(i) else begin
    let m = (i+j) / 2 in
    let (s1,s2) = (| sum a i m, sum a m j |) in
    s1 + s2
 end
```

Efficient use of parallel pairs with granularity control

```
let rec sum a i j =
  if j - i < sequential_cutoff then begin
    let r = ref 0 in
    for k = i to j-1 do
      r := !r + a.(k)
    done;
    !r
  end else begin
    let m = (i+j) / 2 in
    let (s1,s2) = (| sum a i m, sum a m j |) in
    s1 + s2
  end
```

Efficient use of parallel pairs with granularity control

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  if j - i < sequential_cutoff then begin
    let r = ref 0 in
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      r := !r + a.(k)
    done;
    !r
  end else begin
    let m = (i+j) / 2 in
    let (s1,s2) = (| sum a i m, sum a m j |) in
    s1 + s2
  end
```

Generalizable to map-reduce: $f(t[0]) \oplus f(a[1]) \oplus ... \oplus f(a[n-1])$.
(on which condition on $\oplus$?)

Reasoning rule for parallel pairs

$$\frac{\{H_1\}\ t_1\ \{Q_1\} \qquad \{H_2\}\ t_2\ \{Q_2\}}{\{H_1 \star H_2\}\ (|t_1, t_2|)\ \{Q_1 \star Q_2\}}\ \text{PARALLEL}$$

where $Q_1 \star Q_2 \ \equiv\ \lambda(x_1, x_2).\ Q_1\, x_1 \star Q_2\, x_2$

Reasoning rule for parallel pairs

$$\frac{\{H_1\}\ t_1\ \{Q_1\} \qquad \{H_2\}\ t_2\ \{Q_2\}}{\{H_1 \star H_2\}\ (|t_1, t_2|)\ \{Q_1 \star Q_2\}}\ \text{PARALLEL}$$

where $Q_1 \star Q_2 \ \equiv\ \lambda(x_1, x_2).\ Q_1\, x_1 \star Q_2\, x_2$

This rule restricts parallel threads to act on disjoint parts of memory.
(No need for non-interference conditions.)

Parallel rule needs read-only permissions

$$\frac{\{H_1\}\ t_1\ \{Q_1\} \qquad \{H_2\}\ t_2\ \{Q_2\}}{\{H_1 \star H_2\}\ (|t_1, t_2|)\ \{Q_1 \star Q_2\}}\ \textsc{parallel}$$

Compute: $u[a[0]] + u[a[1]] + ... + u[a[n-1]]$.

```
map_reduce (fun x -> u.(x)) 0 (+) 0 n
```

The ownership of the array `u` is needed in both branches.

Parallel rule needs read-only permissions

$$\frac{\{H_1\}\ t_1\ \{Q_1\} \qquad \{H_2\}\ t_2\ \{Q_2\}}{\{H_1 \star H_2\}\ (|t_1, t_2|)\ \{Q_1 \star Q_2\}}\ \text{PARALLEL}$$

Compute: $u[a[0]] + u[a[1]] + \ldots + u[a[n-1]]$.

```
map_reduce (fun x -> u.(x)) 0 (+) 0 n
```

The ownership of the array `u` is needed in both branches.

$$\frac{\{H_1 \star \mathsf{RO}(H_3)\}\ t_1\ \{Q_1\} \qquad \{H_2 \star \mathsf{RO}(H_3)\}\ t_2\ \{Q_2\}}{\{H_1 \star H_2 \star \mathsf{RO}(H_3)\}\ (|t_1, t_2|)\ \{Q_1 \star Q_2\}}\ \text{PARALLEL-RO}$$

Concurrent locks: example

```
let r = ref 0
let s = ref n
let p = create_lock()

let concurrent_step () =
  let () = acquire_lock p in
  incr r;
  decr s;
  release_lock p
```

Concurrent locks: example

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let r = ref 0
let s = ref n
let p = create_lock()

let concurrent_step () =
  let () = acquire_lock p in
  incr r;
  decr s;
  release_lock p
```

Heap predicate $p \rightsquigarrow \mathsf{Lock}\, H$ asserts that lock p protects an invariant H.
Here:

$$p \rightsquigarrow \mathsf{Lock}\, (\exists i.\ (r \mapsto i) \star (s \mapsto n - i))$$

Concurrent locks: specification of operations

Duplicatable representation predicate:

$$p \overset{\mathsf{ro}}{\leadsto} \mathsf{Lock}\, H$$

Operations:

$$\forall H. \quad \{H\}\ (\texttt{create_lock ()})\ \{\lambda p.\ p \overset{\mathsf{ro}}{\leadsto} \mathsf{Lock}\, H\}$$

$$\forall p H. \quad \{p \overset{\mathsf{ro}}{\leadsto} \mathsf{Lock}\, H\}\ (\texttt{acquire_lock p})\ \{\lambda _.\ H\}$$

$$\forall p H.\ \{H \star p \overset{\mathsf{ro}}{\leadsto} \mathsf{Lock}\, H\}\ (\texttt{release_lock p})\ \{\lambda _.\ [\,]\}$$

Concurrent locks: exercise

Exercise: Describe the state at the front of each lines (except 5 and 6). Explicit the instantiation of the existential in the invariant.

```
1    let r = ref 0
2    let s = ref n
3    let p = create_lock()
4
5    let concurrent_step () =
6       let () = acquire_lock p in
7       incr r;
8       decr s;
9       release_lock p
```

Concurrent locks: exercise

Exercise: Describe the state at the front of each lines (except 5 and 6). Explicit the instantiation of the existential in the invariant.

```
1    let r = ref 0
2    let s = ref n
3    let p = create_lock()
4
5    let concurrent_step () =
6       let () = acquire_lock p in
7       incr r;
8       decr s;
9       release_lock p
```

1: $[]$. 2: $r \mapsto 0$. 3: $r \mapsto 0 \star s \mapsto n$.
4: $p \overset{\text{ro}}{\rightsquigarrow} \mathsf{Lock}\,(\exists i.\ (r \mapsto i) \star (s \mapsto n - i))$.
7: $(r \mapsto i) \star (s \mapsto n - i)$. 8: $(r \mapsto i + 1) \star (s \mapsto n - i)$.
9: $(r \mapsto i + 1) \star (s \mapsto n - i - 1)$. Instantiate the invariant with $i + 1$.

Concurrent locks: non-example

```
let r = ref 0
let p = create_lock()

let f () =
  acquire_lock p;
  incr r;
  release_lock p

let () =
  let _ = (| f(), f() |) in
  acquire_lock p;
  assert (!r == 2)
```

Chapter 27

Ghost state

Same non-example

```
let r = ref 0
let p = create_lock()

acquire_lock p;      ||   acquire_lock p;
r := !r + 1;         ||   r := !r + 1;
release_lock p;      ||   release_lock p;

assert (!r == 2);
```

Same non-example

```
let r = ref 0
let p = create_lock()

acquire_lock p;      ||     acquire_lock p;
r := !r + 1;         ||     r := !r + 1;
release_lock p;      ||     release_lock p;

assert (!r == 2);
```

$$p \overset{\mathsf{ro}}{\rightsquigarrow} \mathsf{Lock}\,(\exists n.\ r \mapsto n \star [\ \ \ldots\ ?\ \ldots\ \ ])$$

Same non-example

```
let r = ref 0
let p = create_lock()

acquire_lock p;        ||   acquire_lock p;
r := !r + 1;           ||   r := !r + 1;
release_lock p;        ||   release_lock p;

assert (!r == 2);
```

$$p \overset{\mathsf{ro}}{\leadsto} \mathsf{Lock}\,(\exists n.\ r \mapsto n \star [\ \ldots\ ?\ \ldots\ ])$$

Problem: it is impossible to prove, only with invariants, that this program does not crash (i.e. to prove $\{\mathsf{True}\}$ program $\{\mathsf{True}\}$).

More variables! Ghost variables.

```
let r = ref 0
let r1 = ref 0
let r2 = ref 0
let p = create_lock()
```

```
acquire_lock p;      ||    acquire_lock p;
r := !r + 1;         ||    r := !r + 1;
r1 := !r1 + 1;       ||    r2 := !r2 + 1;
release_lock p;      ||    release_lock p;
```

```
assert (!r == 2);
```

More variables! Ghost variables.

```
let r = ref 0
let r1 = ref 0
let r2 = ref 0
let p = create_lock()

acquire_lock p;      ||    acquire_lock p;
r := !r + 1;         ||    r := !r + 1;
r1 := !r1 + 1;       ||    r2 := !r2 + 1;
release_lock p;      ||    release_lock p;

assert (!r == 2);
```

Exercise: Give a lock invariant that allows proving $\{\text{True}\}$ program $\{\text{True}\}$

More variables! Ghost variables.

```
let r = ref 0
let r1 = ref 0
let r2 = ref 0
let p = create_lock()
```

```
acquire_lock p;      ||    acquire_lock p;
r := !r + 1;         ||    r := !r + 1;
r1 := !r1 + 1;       ||    r2 := !r2 + 1;
release_lock p;      ||    release_lock p;
```

```
assert (!r == 2);
```

Exercise: Give a lock invariant that allows proving $\{\text{True}\}$ program $\{\text{True}\}$

$$p \overset{\text{ro}}{\rightsquigarrow} \mathsf{Lock}\,(\exists n, n_1, n_2.\ (r \mapsto n) \star (r_1 \overset{1/2}{\mapsto} n_1) \star (r_2 \overset{1/2}{\mapsto} n_2) \star [n = n_1 + n_2])$$

More variables! Ghost variables.

```
let r = ref 0
let r1 = ref 0
let r2 = ref 0
let p = create_lock()
```

```
acquire_lock p;      ||   acquire_lock p;
r := !r + 1;         ||   r := !r + 1;
r1 := !r1 + 1;       ||   r2 := !r2 + 1;
release_lock p;      ||   release_lock p;
```

```
assert (!r == 2);
```

Exercise: Give a lock invariant that allows proving {True} program {True}

$$p \overset{\text{ro}}{\rightsquigarrow} \mathsf{Lock}\,(\exists n, n_1, n_2.\ (r \mapsto n) \star (r_1 \overset{1/2}{\mapsto} n_1) \star (r_2 \overset{1/2}{\mapsto} n_2) \star [n = n_1 + n_2])$$

Proof

$$H \equiv \exists n, n_1, n_2.\ (r \mapsto n) \star (r_1 \overset{1/2}{\mapsto} n_1) \star (r_2 \overset{1/2}{\mapsto} n_2) \star [n = n_1 + n_2]$$

```
let r = ref 0
let r1 = ref 0
let r2 = ref 0
```

$\{(r \mapsto 0) * (r_1 \mapsto 0) * (r_2 \mapsto 0)\}$

Proof

$$H \equiv \exists n, n_1, n_2.\ (r \mapsto n) \star (r_1 \overset{1/2}{\mapsto} n_1) \star (r_2 \overset{1/2}{\mapsto} n_2) \star [n = n_1 + n_2]$$

```
let r = ref 0
let r1 = ref 0
let r2 = ref 0
```

$\{(r \mapsto 0) * (r_1 \mapsto 0) * (r_2 \mapsto 0)\}$

$\{H * (r_1 \overset{1/2}{\mapsto} 0) * (r_2 \overset{1/2}{\mapsto} 0)\}$

Proof

$$H \equiv \exists n, n_1, n_2.\ (r \mapsto n) \star (r_1 \overset{1/2}{\mapsto} n_1) \star (r_2 \overset{1/2}{\mapsto} n_2) \star [n = n_1 + n_2]$$

```
let r = ref 0
let r1 = ref 0
let r2 = ref 0
```

$\{(r \mapsto 0) * (r_1 \mapsto 0) * (r_2 \mapsto 0)\}$

$\{H * (r_1 \overset{1/2}{\mapsto} 0) * (r_2 \overset{1/2}{\mapsto} 0)\}$

```
let p = create_lock()
```

Proof

$$H \equiv \exists n, n_1, n_2.\ (r \mapsto n) \star (r_1 \overset{1/2}{\mapsto} n_1) \star (r_2 \overset{1/2}{\mapsto} n_2) \star [n = n_1 + n_2]$$

```
let r = ref 0
let r1 = ref 0
let r2 = ref 0
```

$\{(r \mapsto 0) * (r_1 \mapsto 0) * (r_2 \mapsto 0)\}$

$\{H * (r_1 \overset{1/2}{\mapsto} 0) * (r_2 \overset{1/2}{\mapsto} 0)\}$

```
let p = create_lock()
```

$\{p \overset{\text{ro}}{\rightsquigarrow} \text{Lock}\, H * (r_1 \overset{1/2}{\mapsto} 0) * (r_2 \overset{1/2}{\mapsto} 0)\}$

Proof

$$H \equiv \exists n, n_1, n_2.\ (r \mapsto n) \star (r_1 \overset{1/2}{\mapsto} n_1) \star (r_2 \overset{1/2}{\mapsto} n_2) \star [n = n_1 + n_2]$$

```
let r = ref 0
let r1 = ref 0
let r2 = ref 0
```

$\{(r \mapsto 0) * (r_1 \mapsto 0) * (r_2 \mapsto 0)\}$

$\{H * (r_1 \overset{1/2}{\mapsto} 0) * (r_2 \overset{1/2}{\mapsto} 0)\}$

```
let p = create_lock()
```

$\{p \overset{\mathsf{ro}}{\rightsquigarrow} \mathsf{Lock}\, H * (r_1 \overset{1/2}{\mapsto} 0) * (r_2 \overset{1/2}{\mapsto} 0)\}$

$\{((r_1 \overset{1/2}{\mapsto} 0) * p \overset{\mathsf{ro}}{\rightsquigarrow} \mathsf{Lock}\, H) * ((r_2 \overset{1/2}{\mapsto} 0) * p \overset{\mathsf{ro}}{\rightsquigarrow} \mathsf{Lock}\, H)\}$

Proof

$$H \equiv \exists n, n_1, n_2.\ (r \mapsto n) \star (r_1 \overset{1/2}{\mapsto} n_1) \star (r_2 \overset{1/2}{\mapsto} n_2) \star [n = n_1 + n_2]$$

```
let r = ref 0
let r1 = ref 0
let r2 = ref 0
```

$\{(r \mapsto 0) * (r_1 \mapsto 0) * (r_2 \mapsto 0)\}$

$\{H * (r_1 \overset{1/2}{\mapsto} 0) * (r_2 \overset{1/2}{\mapsto} 0)\}$

```
let p = create_lock()
```

$\{p \overset{\text{ro}}{\rightsquigarrow} \text{Lock}\, H * (r_1 \overset{1/2}{\mapsto} 0) * (r_2 \overset{1/2}{\mapsto} 0)\}$

$\{((r_1 \overset{1/2}{\mapsto} 0) * p \overset{\text{ro}}{\rightsquigarrow} \text{Lock}\, H) * ((r_2 \overset{1/2}{\mapsto} 0) * p \overset{\text{ro}}{\rightsquigarrow} \text{Lock}\, H)\}$

	||	
$\{(r_1 \overset{1/2}{\mapsto} 0 * p \overset{\text{ro}}{\rightsquigarrow} \text{Lock}\, H)\}$	||	$\{(r_2 \overset{1/2}{\mapsto} 0 * p \overset{\text{ro}}{\rightsquigarrow} \text{Lock}\, H)\}$
`acquire_lock p;`	||	`acquire_lock p;`
`r := !r + 1;`	||	`r := !r + 1;`
`...`	||	`...`

Left thread

$$H \equiv \exists n, n_1, n_2.\ (r \mapsto n) \star (r_1 \overset{1/2}{\mapsto} n_1) \star (r_2 \overset{1/2}{\mapsto} n_2) \star [n = n_1 + n_2]$$

$\{(r_1 \overset{1/2}{\mapsto} 0) * p \overset{\mathsf{ro}}{\rightsquigarrow} \mathsf{Lock}\, H\}$

```
acquire_lock p;
```

Left thread

$$H \equiv \exists n, n_1, n_2.\ (r \mapsto n) \star (r_1 \overset{1/2}{\mapsto} n_1) \star (r_2 \overset{1/2}{\mapsto} n_2) \star [n = n_1 + n_2]$$

$\{(r_1 \overset{1/2}{\mapsto} 0) * p \overset{\mathsf{ro}}{\leadsto} \mathsf{Lock}\, H\}$

```
acquire_lock p;
```

$\{(r_1 \overset{1/2}{\mapsto} 0) * H)\}$

Left thread

$$H \equiv \exists n, n_1, n_2.\ (r \mapsto n) \star (r_1 \overset{1/2}{\mapsto} n_1) \star (r_2 \overset{1/2}{\mapsto} n_2) \star [n = n_1 + n_2]$$

$\{(r_1 \overset{1/2}{\mapsto} 0) * p \overset{\text{ro}}{\leadsto} \text{Lock}\, H\}$

```
acquire_lock p;
```

$\{(r_1 \overset{1/2}{\mapsto} 0) * H)\}$ so, for some n, n_1, n_2 such that $n = n_1 + n_2$:

$\{(r_1 \overset{1/2}{\mapsto} 0) * (r \mapsto n) * (r_1 \overset{1/2}{\mapsto} n_1) * (r_2 \overset{1/2}{\mapsto} n_2)\}$

```
r := !r + 1;
```

Left thread

$$H \quad \equiv \quad \exists n, n_1, n_2.\ (r \mapsto n) \star (r_1 \overset{1/2}{\mapsto} n_1) \star (r_2 \overset{1/2}{\mapsto} n_2) \star [n = n_1 + n_2]$$

$\{(r_1 \overset{1/2}{\mapsto} 0) * p \overset{\mathsf{ro}}{\rightsquigarrow} \mathsf{Lock}\, H\}$

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$\{(r_1 \overset{1/2}{\mapsto} 0) * (r \mapsto n) * (r_1 \overset{1/2}{\mapsto} n_1) * (r_2 \overset{1/2}{\mapsto} n_2)\}$

```
r := !r + 1;
```

$\{(r_1 \overset{1/2}{\mapsto} 0) * (r \mapsto n + 1) * (r_1 \overset{1/2}{\mapsto} n_1) * (r_2 \overset{1/2}{\mapsto} n_2)\}$

Left thread

$$H \equiv \exists n, n_1, n_2.\ (r \mapsto n) \star (r_1 \overset{1/2}{\mapsto} n_1) \star (r_2 \overset{1/2}{\mapsto} n_2) \star [n = n_1 + n_2]$$

$\{(r_1 \overset{1/2}{\mapsto} 0) * p \overset{\text{ro}}{\rightsquigarrow} \text{Lock}\, H\}$

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$\{(r_1 \overset{1/2}{\mapsto} 0) * H)\}$ so, for some n, n_1, n_2 such that $n = n_1 + n_2$:

$\{(r_1 \overset{1/2}{\mapsto} 0) * (r \mapsto n) * (r_1 \overset{1/2}{\mapsto} n_1) * (r_2 \overset{1/2}{\mapsto} n_2)\}$

```
r := !r + 1;
```

$\{(r_1 \overset{1/2}{\mapsto} 0) * (r \mapsto n + 1) * (r_1 \overset{1/2}{\mapsto} n_1) * (r_2 \overset{1/2}{\mapsto} n_2)\}$

$\{(r_1 \overset{1}{\mapsto} 0) * (r \mapsto n + 1) * (r_2 \overset{1/2}{\mapsto} n_2)\}$

```
r1 := !r1 + 1;
```

Left thread

$$H \equiv \exists n, n_1, n_2.\ (r \mapsto n) \star (r_1 \overset{1/2}{\mapsto} n_1) \star (r_2 \overset{1/2}{\mapsto} n_2) \star [n = n_1 + n_2]$$

$\{(r_1 \overset{1/2}{\mapsto} 0) * p \overset{\mathsf{ro}}{\rightsquigarrow} \mathsf{Lock}\, H\}$

```
acquire_lock p;
```

$\{(r_1 \overset{1/2}{\mapsto} 0) * H)\}$ so, for some n, n_1, n_2 such that $n = n_1 + n_2$:

$\{(r_1 \overset{1/2}{\mapsto} 0) * (r \mapsto n) * (r_1 \overset{1/2}{\mapsto} n_1) * (r_2 \overset{1/2}{\mapsto} n_2)\}$

```
r := !r + 1;
```

$\{(r_1 \overset{1/2}{\mapsto} 0) * (r \mapsto n + 1) * (r_1 \overset{1/2}{\mapsto} n_1) * (r_2 \overset{1/2}{\mapsto} n_2)\}$

$\{(r_1 \overset{1}{\mapsto} 0) * (r \mapsto n + 1) * (r_2 \overset{1/2}{\mapsto} n_2)\}$

```
r1 := !r1 + 1;
```

$\{(r_1 \overset{1}{\mapsto} 1) * (r \mapsto n + 1) * (r_2 \overset{1/2}{\mapsto} n_2)\}$

Left thread

$$H \equiv \exists n, n_1, n_2.\ (r \mapsto n) \star (r_1 \overset{1/2}{\mapsto} n_1) \star (r_2 \overset{1/2}{\mapsto} n_2) \star [n = n_1 + n_2]$$

$\{(r_1 \overset{1/2}{\mapsto} 0) * p \overset{\text{ro}}{\rightsquigarrow} \text{Lock}\, H\}$

```
acquire_lock p;
```

$\{(r_1 \overset{1/2}{\mapsto} 0) * H)\}$ so, for some n, n_1, n_2 such that $n = n_1 + n_2$:

$\{(r_1 \overset{1/2}{\mapsto} 0) * (r \mapsto n) * (r_1 \overset{1/2}{\mapsto} n_1) * (r_2 \overset{1/2}{\mapsto} n_2)\}$

```
r := !r + 1;
```

$\{(r_1 \overset{1/2}{\mapsto} 0) * (r \mapsto n + 1) * (r_1 \overset{1/2}{\mapsto} n_1) * (r_2 \overset{1/2}{\mapsto} n_2)\}$

$\{(r_1 \overset{1}{\mapsto} 0) * (r \mapsto n + 1) * (r_2 \overset{1/2}{\mapsto} n_2)\}$

```
r1 := !r1 + 1;
```

$\{(r_1 \overset{1}{\mapsto} 1) * (r \mapsto n + 1) * (r_2 \overset{1/2}{\mapsto} n_2)\}$

$\{(r_1 \overset{1/2}{\mapsto} 1) * (r \mapsto n + 1) * (r_1 \overset{1/2}{\mapsto} 1) * (r_2 \overset{1/2}{\mapsto} n_2)\}$

Left thread

$$H \quad \equiv \quad \exists n, n_1, n_2.\ (r \mapsto n) \star (r_1 \overset{1/2}{\mapsto} n_1) \star (r_2 \overset{1/2}{\mapsto} n_2) \star [n = n_1 + n_2]$$

$\{(r_1 \overset{1/2}{\mapsto} 0) * p \overset{\mathsf{ro}}{\leadsto} \mathsf{Lock}\, H\}$

```
acquire_lock p;
```

$\{(r_1 \overset{1/2}{\mapsto} 0) * H)\}$ so, for some n, n_1, n_2 such that $n = n_1 + n_2$:

$\{(r_1 \overset{1/2}{\mapsto} 0) * (r \mapsto n) * (r_1 \overset{1/2}{\mapsto} n_1) * (r_2 \overset{1/2}{\mapsto} n_2)\}$

```
r := !r + 1;
```

$\{(r_1 \overset{1/2}{\mapsto} 0) * (r \mapsto n + 1) * (r_1 \overset{1/2}{\mapsto} n_1) * (r_2 \overset{1/2}{\mapsto} n_2)\}$

$\{(r_1 \overset{1}{\mapsto} 0) * (r \mapsto n + 1) * (r_2 \overset{1/2}{\mapsto} n_2)\}$

```
r1 := !r1 + 1;
```

$\{(r_1 \overset{1}{\mapsto} 1) * (r \mapsto n + 1) * (r_2 \overset{1/2}{\mapsto} n_2)\}$

$\{(r_1 \overset{1/2}{\mapsto} 1) * (r \mapsto n + 1) * (r_1 \overset{1/2}{\mapsto} 1) * (r_2 \overset{1/2}{\mapsto} n_2)\}$

$\{(r_1 \overset{1/2}{\mapsto} 1) * H\}$

```
release_lock p;
```

Left thread

$$H \equiv \exists n, n_1, n_2.\ (r \mapsto n) \star (r_1 \overset{1/2}{\mapsto} n_1) \star (r_2 \overset{1/2}{\mapsto} n_2) \star [n = n_1 + n_2]$$

$\{(r_1 \overset{1/2}{\mapsto} 0) * p \overset{\mathsf{ro}}{\leadsto} \mathsf{Lock}\, H\}$

```
acquire_lock p;
```

$\{(r_1 \overset{1/2}{\mapsto} 0) * H)\}$ so, for some n, n_1, n_2 such that $n = n_1 + n_2$:

$\{(r_1 \overset{1/2}{\mapsto} 0) * (r \mapsto n) * (r_1 \overset{1/2}{\mapsto} n_1) * (r_2 \overset{1/2}{\mapsto} n_2)\}$

```
r := !r + 1;
```

$\{(r_1 \overset{1/2}{\mapsto} 0) * (r \mapsto n + 1) * (r_1 \overset{1/2}{\mapsto} n_1) * (r_2 \overset{1/2}{\mapsto} n_2)\}$

$\{(r_1 \overset{1}{\mapsto} 0) * (r \mapsto n + 1) * (r_2 \overset{1/2}{\mapsto} n_2)\}$

```
r1 := !r1 + 1;
```

$\{(r_1 \overset{1}{\mapsto} 1) * (r \mapsto n + 1) * (r_2 \overset{1/2}{\mapsto} n_2)\}$

$\{(r_1 \overset{1/2}{\mapsto} 1) * (r \mapsto n + 1) * (r_1 \overset{1/2}{\mapsto} 1) * (r_2 \overset{1/2}{\mapsto} n_2)\}$

$\{(r_1 \overset{1/2}{\mapsto} 1) * H\}$

```
release_lock p;
```

$\{(r_1 \overset{1/2}{\mapsto} 1)\}$

Left thread

$$H \equiv \exists n, n_1, n_2.\ (r \mapsto n) \star (r_1 \overset{1/2}{\mapsto} n_1) \star (r_2 \overset{1/2}{\mapsto} n_2) \star [n = n_1 + n_2]$$

$\{(r_1 \overset{1/2}{\mapsto} 0) * p \overset{\text{ro}}{\rightsquigarrow} \mathsf{Lock}\, H\}$

```
acquire_lock p;
```

$\{(r_1 \overset{1/2}{\mapsto} 0) * H)\}$ so, for some n, n_1, n_2 such that $n = n_1 + n_2$:

$\{(r_1 \overset{1/2}{\mapsto} 0) * (r \mapsto n) * (r_1 \overset{1/2}{\mapsto} n_1) * (r_2 \overset{1/2}{\mapsto} n_2)\}$

```
r := !r + 1;
```

$\{(r_1 \overset{1/2}{\mapsto} 0) * (r \mapsto n+1) * (r_1 \overset{1/2}{\mapsto} n_1) * (r_2 \overset{1/2}{\mapsto} n_2)\}$

$\{(r_1 \overset{1}{\mapsto} 0) * (r \mapsto n+1) * (r_2 \overset{1/2}{\mapsto} n_2)\}$

```
r1 := !r1 + 1;
```

$\{(r_1 \overset{1}{\mapsto} 1) * (r \mapsto n+1) * (r_2 \overset{1/2}{\mapsto} n_2)\}$

$\{(r_1 \overset{1/2}{\mapsto} 1) * (r \mapsto n+1) * (r_1 \overset{1/2}{\mapsto} 1) * (r_2 \overset{1/2}{\mapsto} n_2)\}$

$\{(r_1 \overset{1/2}{\mapsto} 1) * H\}$

```
release_lock p;
```

$\{(r_1 \overset{1/2}{\mapsto} 1)\}$ no $p \overset{\text{ro}}{\rightsquigarrow} \mathsf{Lock}\, H$?

Right thread

$$H \equiv \exists n, n_1, n_2.\ (r \mapsto n) \star (r_1 \overset{1/2}{\mapsto} n_1) \star (r_2 \overset{1/2}{\mapsto} n_2) \star [n = n_1 + n_2]$$

$\{(r_2 \overset{1/2}{\mapsto} 0) * p \overset{\text{ro}}{\rightsquigarrow} \mathsf{Lock}\, H\}$

```
acquire_lock p;
```

$\{(r_2 \overset{1/2}{\mapsto} 0) * H)\}$

```
r := !r + 1;
r2 := !r2 + 1;
```

$\{(r_2 \overset{1/2}{\mapsto} 1) * H\}$

```
release_lock p;
```

$\{(r_2 \overset{1/2}{\mapsto} 1)\}$

Finish up

```
let r = ref 0
let r1 = ref 0
let r2 = ref 0
let p = create_lock()
```

Left thread:

$\{(r_1 \overset{1/2}{\mapsto} 0) * p \overset{\mathsf{ro}}{\leadsto} \mathsf{Lock}\, H\}$

```
acquire_lock p;
r := !r + 1;
r1 := !r1 + 1;
release_lock p;
```

$\{(r_1 \overset{1/2}{\mapsto} 1) * p \overset{\mathsf{ro}}{\leadsto} \mathsf{Lock}\, H\}$

Right thread (in parallel, $\parallel$):

$\{(r_2 \overset{1/2}{\mapsto} 0) * p \overset{\mathsf{ro}}{\leadsto} \mathsf{Lock}\, H\}$

```
acquire_lock p;
r := !r + 1;
r2 := !r2 + 1;
release_lock p;
```

$\{(r_2 \overset{1/2}{\mapsto} 1) * p \overset{\mathsf{ro}}{\leadsto} \mathsf{Lock}\, H\}$

$\{(r_1 \overset{1/2}{\mapsto} 1) * (r_2 \overset{1/2}{\mapsto} 1) * (r \mapsto n) * (r_1 \overset{1/2}{\mapsto} n_1) * (r_2 \overset{1/2}{\mapsto} n_2) * [n = n_1 + n_2]\}$

$\{(r_1 \mapsto 1) * (r_2 \mapsto 1) * (r \mapsto n) * [n = 1 + 1]\}$

```
assert (!r == 2);
```

$$p \overset{\mathsf{ro}}{\leadsto} \mathsf{Lock}\,(\exists n, n_1, n_2.\ (r \mapsto n) \star (r_1 \overset{1/2}{\mapsto} n_1) \star (r_2 \overset{1/2}{\mapsto} n_2) \star [n = n_1 + n_2])$$

Question: Would replacing "$1/2$" with "ro" work?

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- allocation of ghost state: for some a any "Resource Algebra":

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- validity in RA's e.g. $\mathsf{valid}((r_2 \mapsto 1) \cdot (r_2 \mapsto n_2)) \;\Rightarrow\; n_2 = 1$

Conclusion

Program verification using Separation Logic gives you:

- Expressiveness: tree-shaped structures, and structures with sharing
- Expressiveness: effectful, first-class functions, with local state
- Expressiveness: concurrency, ghost state
- Modularity: most-general specifications
- Modularity: composable representation predicates
- Abstraction: existential quantification of intermediate pointers
- Abstraction: existential quantification of invariants
- Practice: formalization in Coq of all heap predicates
- Practice: characteristic formulas for reasoning rules
- Practice: more advanced separation logic
  `https://iris-project.org/#learning`