Separation Logic 3/4

Jean-Marie Madiot

Inria Paris

February 2, 2024

part 3/4 : most material from Arthur Charguéraud

Chapter 12

Loops in Separation Logic

Verification of a for-loop

```
let facto n =
  let r = ref 1 in
  for i = 2 to n do
    let v = !r in
    r := v * i;
  done;
  !r
```

Verification of a for-loop

```
let facto n =
  let r = ref 1 in
  for i = 2 to n do
    let v = !r in
    r := v * i;
  done;
  !r
```

Before the loop:

$$r \mapsto 1$$

At each iteration:

$$\text{from} \quad r \mapsto (i-1)! \quad \text{to} \quad r \mapsto i!$$

After the loop:

$$r \mapsto n!$$

Verification of a for-loop

```
let facto n =
  let r = ref 1 in
  for i = 2 to n do
    let v = !r in
    r := v * i;
  done;
  !r
```

Before the loop:

$$r \mapsto 1$$

At each iteration:

$$\text{from} \quad r \mapsto (i-1)! \quad \text{to} \quad r \mapsto i!$$

After the loop:

$$r \mapsto n!$$

Loop invariant $(I : \text{int} \to \text{Hprop})$ that applies for any $i \in [2, n+1]$:

$$I\, i \quad \equiv \quad r \mapsto (i-1)!$$

Reasoning rule for for-loops

Reasoning rule for the case $a \leqslant b$:

$$\frac{\begin{array}{c} H \rhd I\, a \\ \forall i \in [a, b].\quad \{I\, i\}\ t\ \{\lambda\_.\, I\, (i+1)\} \\ I\, (b+1) \rhd Q\, () \end{array}}{\{H\}\ (\mathsf{for}\, i = a\, \mathsf{to}\, b\, \mathsf{do}\, t)\ \{Q\}}$$

Reasoning rule for for-loops

Reasoning rule for the case $a \leqslant b$:

$$\frac{\begin{array}{c} H \rhd I\, a \\ \forall i \in [a, b].\quad \{I\, i\}\ t\ \{\lambda\_.\, I\, (i+1)\} \\ I\, (b+1) \rhd Q\, () \end{array}}{\{H\}\ (\mathsf{for}\, i = a\, \mathsf{to}\, b\, \mathsf{do}\, t)\ \{Q\}}$$

General rule, also covering the case $a > b$:

$$\frac{\begin{array}{c} H \rhd I\, a \\ \forall i \in [a, b].\quad \{I\, i\}\ t\ \{\lambda\_.\, I\, (i+1)\} \\ I\, (\mathsf{max}\, a\, (b+1)) \rhd Q\, () \end{array}}{\{H\}\ (\mathsf{for}\, i = a\, \mathsf{to}\, b\, \mathsf{do}\, t)\ \{Q\}}$$

Reasoning rule for while loops: partial correctness

The loop invariant I describes the state between every iterations.
The post-condition J describes the state after the evaluation of t_1.

$$\frac{H \rhd I \qquad \{I\}\ t_1\ \{J\} \qquad \{J\,\mathsf{true}\}\ t_2\ \{\lambda\_.\ I\} \qquad J\,\mathsf{false} \rhd Q\,()}{\{H\}\ (\mathsf{while}\,t_1\,\mathsf{do}\,t_2)\ \{Q\}}$$

where $(I : \mathsf{Hprop})$ and $(J : \mathsf{bool} \to \mathsf{Hprop})$.

Reasoning rule for while loops: partial correctness

The loop invariant I describes the state between every iterations.
The post-condition J describes the state after the evaluation of t_1.

$$\frac{H \rhd I \qquad \{I\}\ t_1\ \{J\} \qquad \{J\,\mathsf{true}\}\ t_2\ \{\lambda\_.\ I\} \qquad J\,\mathsf{false} \rhd Q\,()}{\{H\}\ (\mathsf{while}\,t_1\,\mathsf{do}\,t_2)\ \{Q\}}$$

where $(I : \mathsf{Hprop})$ and $(J : \mathsf{bool} \to \mathsf{Hprop})$.

For total correctness: parameterize the invariant with a measure.

Reasoning rule for while loops

We focus on a different approach that:

- inherently supports total correctness;
- allows to apply frame during iterations.

Reasoning rule for while loops

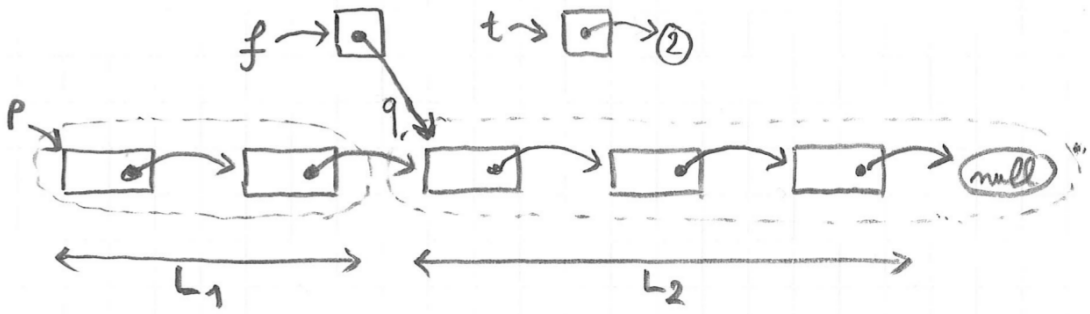
We focus on a different approach that:

- inherently supports total correctness;
- allows to apply frame during iterations.

Prove a triple $\{H\}\ (\mathsf{while}\, t_1\, \mathsf{do}\, t_2)\ \{Q\}$ by induction, using:

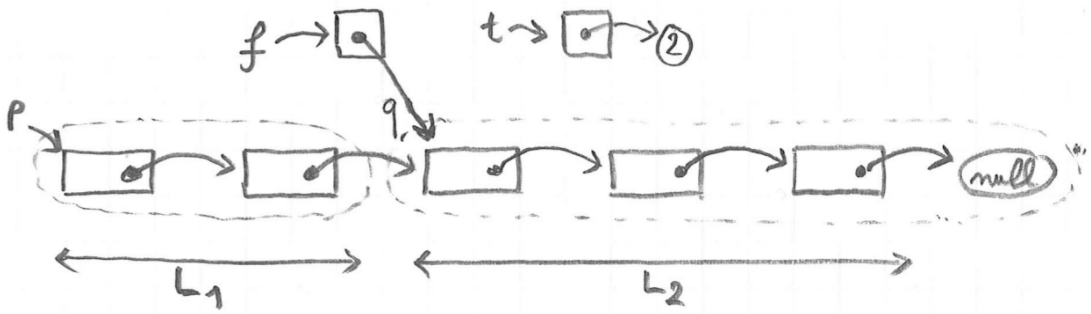
$$\frac{\{H\}\ (\mathsf{if}\, t_1\, \mathsf{then}\, (t_2\, ;\, (\mathsf{while}\, t_1\, \mathsf{do}\, t_2))\, \mathsf{else}\, ())\ \{Q\}}{\{H\}\ (\mathsf{while}\, t_1\, \mathsf{do}\, t_2)\ \{Q\}}$$

Length with a while loop




```
let mlength (p:'a cell) =
  let t = ref 0 in
  let f = ref p in
  while !f != null do
    incr t;
    f := (!f).tl;
  done;
  !t
```

Length with a while loop: induction



We prove by induction on L_2 that for any n and q:

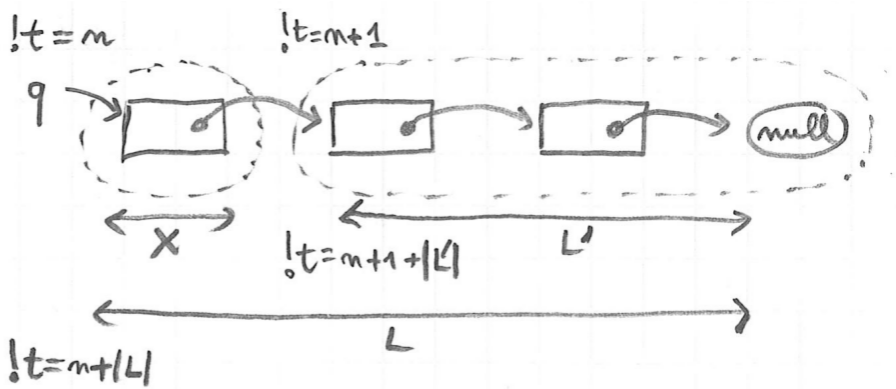
$$\{q \rightsquigarrow \mathsf{MList}\, L_2 \;*\; f \mapsto q \;*\; t \mapsto n\}$$
$$(\texttt{while !f != null do incr t; f := (!f).tl; done})$$
$$\{q \rightsquigarrow \mathsf{MList}\, L_2 \;*\; f \mapsto \mathsf{null} \;*\; t \mapsto (n + \mathsf{length}\, L_2)\}$$

The loop unfolds to:

```
if !f != null
  then (incr t; f := (!f).tl; while .. do .. done)
  else ()
```

Exercise: describe the frame process in the induction for `mlength`.

Length with a while loop: frame process



$q \rightsquigarrow \mathsf{MList}\, L_2$		$*$	$f \mapsto q$	$*$	$t \mapsto n$	begin
$q \mapsto \{\!\|x; q'\|\!\} * q' \rightsquigarrow \mathsf{MList}\, L_2'$		$*$	$f \mapsto q$	$*$	$t \mapsto n$	unfold
$q \mapsto \{\!\|x; q'\|\!\} * q' \rightsquigarrow \mathsf{MList}\, L_2'$		$*$	$f \mapsto q$	$*$	$t \mapsto n+1$	increment
$\underline{q \mapsto \{\!\|x; q'\|\!\}} * q' \rightsquigarrow \mathsf{MList}\, L_2'$		$*$	$f \mapsto q'$	$*$	$t \mapsto n+1$	shift head
$q \mapsto \{\!\|x; q'\|\!\} * q' \rightsquigarrow \mathsf{MList}\, L_2'$		$*$	$f \mapsto \mathsf{null}$	$*$	$t \mapsto n+1+\|L_2'\|$	frame+Ind.hyp.
$q \rightsquigarrow \mathsf{MList}\, L_2$		$*$	$f \mapsto \mathsf{null}$	$*$	$t \mapsto n+\|L_2\|$	fold

Chapter 13

Aliasing and local state

Functions with aliasing: swap

```
let swap r s =
  let a = !r in
  let b = !s in
  r := b;
  s := a
```

Exercise: Find three useful specifications for `swap`:

1. a specification for non-aliased (distinct) arguments,
2. a specification for aliased (equal) arguments,
3. a most-general specification, stated using iterated conjunction (or another construct from Course 2).

Functions with aliasing: 3 specifications for swap

Specification 1:

$$\forall r s n m.\ \{(r \mapsto n) * (s \mapsto m)\}\ (\texttt{swap r s})\ \{\lambda\_.\ (r \mapsto m) * (s \mapsto n)\}$$

Functions with aliasing: 3 specifications for swap

Specification 1:

$$\forall rsnm.\ \{(r \mapsto n) * (s \mapsto m)\}\ (\texttt{swap r s})\ \{\lambda\_.\ (r \mapsto m) * (s \mapsto n)\}$$

Specification 2:

$$\forall rsn.\ \{\ulcorner r = s \urcorner * (r \mapsto n)\}\ (\texttt{swap r s})\ \{\lambda\_.\ r \mapsto n\}$$

or simply:

$$\forall rn.\ \{r \mapsto n\}\ (\texttt{swap r r})\ \{\lambda\_.\ r \mapsto n\}$$

Functions with aliasing: 3 specifications for swap

Specification 1:

$$\forall rsnm.\ \{(r \mapsto n) * (s \mapsto m)\}\ (\texttt{swap r s})\ \{\lambda\_.\ (r \mapsto m) * (s \mapsto n)\}$$

Specification 2:

$$\forall rsn.\ \{\ulcorner r = s \urcorner * (r \mapsto n)\}\ (\texttt{swap r s})\ \{\lambda\_.\ r \mapsto n\}$$

or simply:

$$\forall rn.\ \{r \mapsto n\}\ (\texttt{swap r r})\ \{\lambda\_.\ r \mapsto n\}$$

Specification 3:

$$\begin{aligned}\forall rsM.\ r, s \in \mathsf{dom}\, M \ \Rightarrow\ & \{\circledast_{(p,n)\in M}\ p \mapsto n\} \\ & (\texttt{swap r s}) \\ & \{\lambda\_.\ \circledast_{(p,n)\in (M[r:=M[s]][s:=M[r]])}\ p \mapsto n\}\end{aligned}$$

alternatively with a no-offset version of Cells, $\forall rsM.\ r, s \in \mathsf{dom}\, M \ \Rightarrow$

$$\{\mathsf{Cells}'(M)\}\ (\texttt{swap r s})\ \{\lambda\_.\ \mathsf{Cells}'\ (M[r := M[s]][s := M[r]])\}$$

Function with local state

Exercise: what is the specification of `f` in the following program?

```
let r = ref 3
let f () =
  incr r
```

Then, show that the code below returns 5.

```
f();
f();
!r
```

Function with local state

Exercise: what is the specification of `f` in the following program?

```
let r = ref 3
let f () =
  incr r
```

Then, show that the code below returns 5.

```
f();
f();
!r
```

Specification:

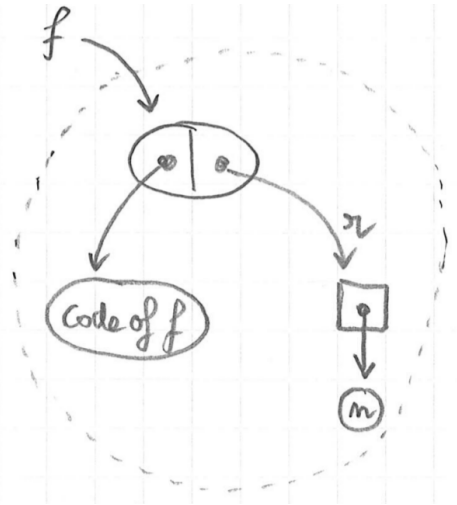
$$\forall n. \quad \{r \mapsto n\}\ (\texttt{f}\ \ ())\ \{\lambda\_.\ r \mapsto n+1\}$$

Successive states:

$$r \mapsto 3 \qquad r \mapsto 4 \qquad r \mapsto 5$$

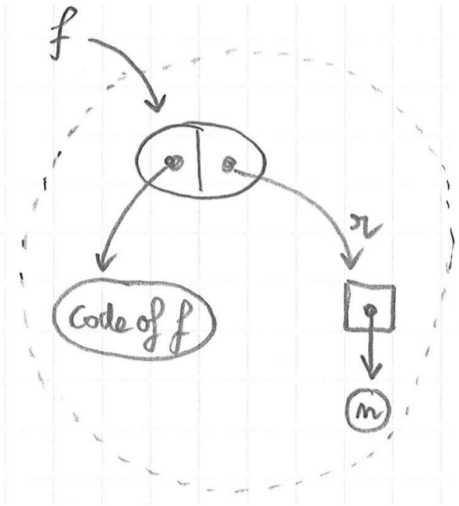
Counter function: code

```
let mkcounter () =
  let r = ref 0 in
  (fun () -> incr r; !r)
```



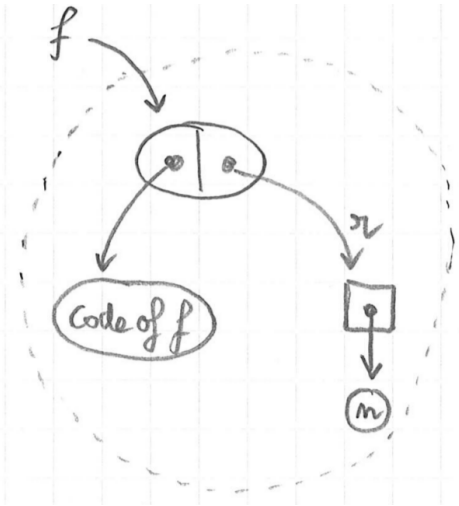

```
let c = mkcounter() in
let x = c() in
let y = c() in
assert (x = 1 && y = 2)
```

Counter function: specification



$$f \rightsquigarrow \mathsf{Count}\, n \;\equiv\; \exists r.\, (r \mapsto n) \\ *\; \ulcorner \forall i.\, \{r \mapsto i\}\; (f\; ()\,)\; \{\lambda x. \ulcorner x = i + 1 \urcorner * (r \mapsto i + 1)\} \urcorner$$

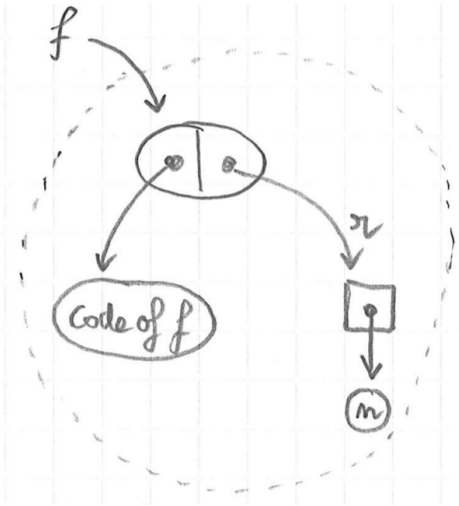
Counter function: specification



$$f \rightsquigarrow \mathsf{Count}\, n \;\equiv\; \exists r.\, (r \mapsto n) \\ *\; \ulcorner \forall i.\, \{r \mapsto i\}\; (f\; ()\,)\; \{\lambda x. \ulcorner x = i + 1 \urcorner * (r \mapsto i + 1)\} \urcorner$$

Exercise: specify a counter function, only in terms of $f \rightsquigarrow \mathsf{Count}\, n$.

Counter function: specification



$$f \rightsquigarrow \mathsf{Count}\, n \;\equiv\; \exists r.\, (r \mapsto n) \\ *\; \ulcorner \forall i.\, \{r \mapsto i\}\; (f\; ())\; \{\lambda x.\, \ulcorner x = i + 1 \urcorner * (r \mapsto i + 1)\} \urcorner$$

Exercise: specify a counter function, only in terms of $f \rightsquigarrow \mathsf{Count}\, n$.

$$\{\ulcorner \urcorner\}\; (\texttt{mkcounter}\,())\; \{\lambda f.\; f \rightsquigarrow \mathsf{Count}\, 0\}$$

$$\forall f i.\quad \{f \rightsquigarrow \mathsf{Count}\, i\}\; (f\; ())\; \{\lambda x.\, \ulcorner x = i + 1 \urcorner * f \rightsquigarrow \mathsf{Count}\, (i + 1)\}$$

Chapter 14

Basic higher-order functions

Apply

```
let apply f x =
  f x
```

Specification:

$$\forall f\, x\, H\, Q. \qquad \{H\}\ (f\ x)\ \{Q\}$$
$$\Rightarrow \quad \{H\}\ (\texttt{apply}\ f\ x)\ \{Q\}$$

Apply

```
let apply f x =
  f x
```

Specification:

$$\begin{array}{lll} \forall f\, x\, H\, Q. & & \{H\}\ (f\ x)\ \{Q\} \\ & \Rightarrow & \{H\}\ (\texttt{apply}\ f\ x)\ \{Q\} \end{array}$$

This is equivalent to the form below, which involves nested triples:

$$\forall f\, x\, H\, Q.\quad \{H\ *\ \ulcorner \{H\}\ (f\ x)\ \{Q\} \urcorner\}\ (\texttt{apply}\ f\ x)\ \{Q\}$$

Apply on a reference

```
let refapply r f =
  r := f !r
```

Exercise: give two specifications for the function `refapply`.
In the first, assume `f` to be pure, and introduce a predicate $P\,x\,y$.
In the second, assume that `f` also modifies the state from H to H'.

Apply on a reference

```
let refapply r f =
  r := f !r
```

Exercise: give two specifications for the function `refapply`.
In the first, assume `f` to be pure, and introduce a predicate $P\,x\,y$.
In the second, assume that `f` also modifies the state from H to H'.

$$\forall r\,f\,x\,P. \quad \{\ulcorner\urcorner\}\ (f\,x)\ \{\lambda y.\ \ulcorner P\,x\,y\urcorner\}$$
$$\Rightarrow \quad \{r \mapsto x\}\ (\texttt{refapply}\,r\,f)\ \{\lambda\_.\ \exists y.\ \ulcorner P\,x\,y\urcorner * r \mapsto y\}$$

Apply on a reference

```
let refapply r f =
  r := f !r
```

Exercise: give two specifications for the function `refapply`.
In the first, assume `f` to be pure, and introduce a predicate $P\,x\,y$.
In the second, assume that `f` also modifies the state from H to H'.

$$\forall r\,f\,x\,P. \quad \{\ulcorner\urcorner\}\ (f\,x)\ \{\lambda y.\ \ulcorner P\,x\,y\urcorner\}$$
$$\Rightarrow \quad \{r \mapsto x\}\ (\texttt{refapply}\,r\,f)\ \{\lambda\_.\ \exists y.\ \ulcorner P\,x\,y\urcorner * r \mapsto y\}$$

$$\forall r\,f\,x\,H\,H'\,P. \quad \{H\}\ (f\,x)\ \{\lambda y.\ \ulcorner P\,x\,y\urcorner * H'\}$$
$$\Rightarrow \quad \{(r \mapsto x) * H\}\ (\texttt{refapply}\,r\,f)\ \{\lambda\_.\ \exists y.\ \ulcorner P\,x\,y\urcorner * (r \mapsto y) * H'\}$$

Function twice

```
let twice f =
  f(); f()
```

Specification:

$$\begin{aligned} \forall f\, H'\, Q. \quad & \{H\}\ (f\ ())\ \{\lambda\_.\ H'\} \\ \wedge \quad & \{H'\}\ (f\ ())\ \{Q\} \\ \Rightarrow \quad & \{H\}\ (\texttt{twice}\ f)\ \{Q\} \end{aligned}$$

Function repeat

```
let repeat n f =
  for i = 0 to n-1 do
    f()
  done
```

Exercise: specify `repeat`, using an invariant I, of type int $\rightarrow$ Hprop.

Function repeat

```
let repeat n f =
  for i = 0 to n-1 do
    f()
  done
```

Exercise: specify `repeat`, using an invariant I, of type int $\to$ Hprop.

$$\begin{array}{lll} \forall n f I. & & (\forall i \in [0, n).\quad \{I\,i\}\ (f\ ())\ \{\lambda\_.\ I\,(i+1)\}) \\ & \Rightarrow & \{I\,0\}\ (\texttt{repeat}\,n\,f)\ \{\lambda\_.\ I\,n\} \end{array}$$

The premise consists of a family of hypotheses describing the behavior of applications of f to particular arguments.

Chapter 15

Higher order iteration

Iteration over a pure list

For pedagogical purposes, "pure lists" live outside the heap and need no representation predicate. (In practice, most (but not all) lists are allocated on the heap.)

```
let rec iter f l =
  match l with
  | [] -> ()
  | x::t -> f x; iter f t
```

Exercise: specify `iter`, using an invariant I, of type list $\alpha \to$ Hprop.

Iteration over a pure list

For pedagogical purposes, "pure lists" live outside the heap and need no representation predicate. (In practice, most (but not all) lists are allocated on the heap.)

```
let rec iter f l =
  match l with
  | [] -> ()
  | x::t -> f x; iter f t
```

Exercise: specify `iter`, using an invariant I, of type list $\alpha \to$ Hprop.

$$\begin{array}{rl} \forall f l I. & \left(\forall x k.\ \{I\,k\}\ (f\,x)\ \{\lambda\_.\ I\,(k \& x)\}\right) \\ \Rightarrow & \{I\,\mathsf{nil}\}\ (\texttt{iter}\,f\,l)\ \{\lambda\_.\ I\,l\} \end{array}$$

where $k \& x \equiv k +\!\!+ (x :: \mathsf{nil})$.

Length using iter

$$\begin{array}{rl} & (\forall x k.\ \{I\ k\}\ (f\ x)\ \{\lambda\_.\ I\ (k \& x)\}) \\ \Rightarrow & \{I\ \mathsf{nil}\}\ (\mathsf{iter}\ f\ l)\ \{\lambda\_.\ I\ l\} \end{array}$$

```
let length l =
  let r = ref 0 in
  iter (fun x -> incr r) l;
  !r
```

Exercise: give the instantiatiation of the invariant I for `iter`; then, write the specialization of the specification of `iter` to I and to `(fun x -> incr r)`; finally, check that the premise is provable.

Length using iter

$$\begin{array}{rl} & \big(\forall xk.\ \{I\,k\}\ (f\,x)\ \{\lambda\_.\ I\,(k\&x)\}\big) \\ \Rightarrow & \{I\,\mathsf{nil}\}\ (\mathsf{iter}\,f\,l)\ \{\lambda\_.\ I\,l\} \end{array}$$

```
let length l =
  let r = ref 0 in
  iter (fun x -> incr r) l;
  !r
```

Exercise: give the instantiatiation of the invariant I for `iter`; then, write the specialization of the specification of `iter` to I and to `(fun x -> incr r)`; finally, check that the premise is provable.

Invariant: $I \equiv \lambda k.\ r \mapsto |k|$.

$$\begin{array}{rl} & \big(\forall xk.\ \{r \mapsto |k|\}\ (\texttt{incr r})\ \{\lambda\_.\ r \mapsto |k| + 1\}\big) \\ \Rightarrow & \{r \mapsto 0\}\ (\mathsf{iter}\,f\,l)\ \{\lambda\_.\ r \mapsto |l|\} \end{array}$$

Sum using iter

$$\begin{array}{rl} & (\forall x k.\ \{I\ k\}\ (f\ x)\ \{\lambda\_.\ I\ (k \& x)\}) \\ \Rightarrow & \{I\ \mathsf{nil}\}\ (\mathsf{iter}\ f\ l)\ \{\lambda\_.\ I\ l\} \end{array}$$

```
let sum l =
  let r = ref 0 in
  iter (fun x -> r := !r + x) l;
  !r
```

Exercise: give the invariant I involved in the above call to `iter`.

Sum using iter

$$\begin{array}{rl} & (\forall x k.\ \{I\ k\}\ (f\ x)\ \{\lambda\_.\ I\ (k \& x)\}) \\ \Rightarrow & \{I\ \mathsf{nil}\}\ (\mathsf{iter}\ f\ l)\ \{\lambda\_.\ I\ l\} \end{array}$$

```
let sum l =
  let r = ref 0 in
  iter (fun x -> r := !r + x) l;
  !r
```

Exercise: give the invariant I involved in the above call to `iter`.

$$I \ \equiv\ \lambda k.\ r \mapsto \mathsf{Sum}\ k$$

where:

$$\mathsf{Sum}\ k \ \equiv\ \mathsf{Fold}\ (+)\ 0\ k$$

Constraints over the items

$$\begin{array}{rl} & \left(\forall x k.\ \{I\ k\}\ (f\ x)\ \{\lambda\_.\ I\ (k \& x)\}\right) \\ \Rightarrow & \{I\ \mathsf{nil}\}\ (\mathsf{iter}\ f\ l)\ \{\lambda\_.\ I\ l\} \end{array}$$

Given a list $x_1 :: x_2 :: ... :: x_n ::$ nil, let us compute:

$$\frac{10}{x_1} + \frac{10}{x_2} + ... + \frac{10}{x_n}$$

```
iter (fun x -> r := !r + 10 / x) [2; -3; 4]
```

Constraints over the items

$$\begin{array}{rl} & \left(\forall xk.\ \{I\ k\}\ (f\ x)\ \{\lambda\_.\ I\ (k \& x)\}\right) \\ \Rightarrow & \{I\ \mathsf{nil}\}\ (\mathsf{iter}\ f\ l)\ \{\lambda\_.\ I\ l\} \end{array}$$

Given a list $x_1 :: x_2 :: ... :: x_n :: \mathsf{nil}$, let us compute:

$$\frac{10}{x_1} + \frac{10}{x_2} + ... + \frac{10}{x_n}$$

```
iter (fun x -> r := !r + 10 / x) [2; -3; 4]
```

The above specification of `iter` is too weak. More general specification:

$$\begin{array}{lll} \forall f I l. & & \left(\forall xk.\ x \in l \Rightarrow \{I\ k\}\ (f\ x)\ \{\lambda\_.\ I\ (k \& x)\}\right) \\ & \Rightarrow & \{I\ \mathsf{nil}\}\ (\mathsf{iter}\ f\ l)\ \{\lambda\_.\ I\ l\} \end{array}$$

Constraints over the items, in order

$$\forall f\, I\, l. \qquad \big(\forall x\, k.\ x \in l \Rightarrow \{I\, k\}\ (f\, x)\ \{\lambda\_.\ I\ (k \& x)\}\big)$$
$$\Rightarrow \quad \{I\ \mathsf{nil}\}\ (\mathsf{iter}\ f\ l)\ \{\lambda\_.\ I\ l\}$$

Given a list $x_1 :: x_2 :: \ldots :: x_n :: \mathsf{nil}$, let us compute:

$$\cfrac{10}{\cfrac{10}{\cfrac{10}{0+x_1}+x_2}\ddots+x_n}$$

```
iter (fun x -> r := 10 / (!r + x)) [2; -3; 4]
```

Constraints over the items, in order

$$\forall f\, I\, l. \qquad \big(\forall x k.\ x \in l \Rightarrow \{I\, k\}\ (f\, x)\ \{\lambda\_.\ I\ (k \& x)\}\big)$$
$$\Rightarrow \quad \{I\ \mathsf{nil}\}\ (\mathsf{iter}\, f\, l)\ \{\lambda\_.\ I\, l\}$$

Given a list $x_1 :: x_2 :: \ldots :: x_n :: \mathsf{nil}$, let us compute:

$$\cfrac{10}{\cfrac{10}{\cfrac{10}{0+x_1}+x_2}\ddots+x_n}$$

```
iter (fun x -> r := 10 / (!r + x)) [2; -3; 4]
```

The above specification of `iter` is too weak. Most-general specification:

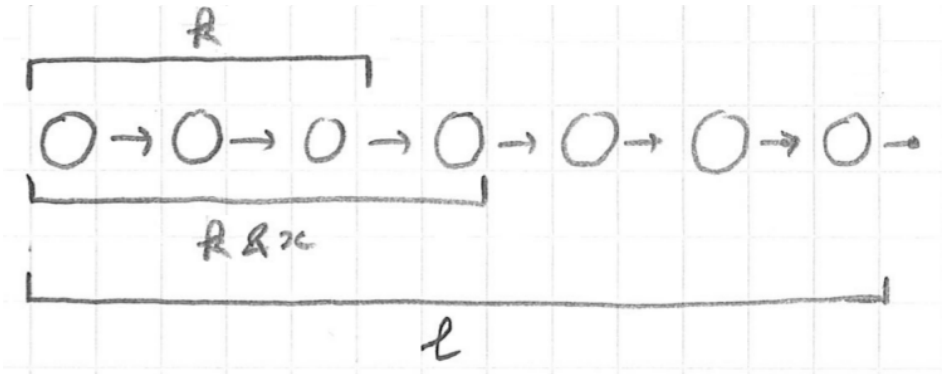
$$\forall f\, I\, l. \qquad \big(\forall x k s.\ l = k +\!\!+ x :: s \Rightarrow \{I\, k\}\ (f\, x)\ \{\lambda\_.\ I\ (k \& x)\}\big)$$
$$\Rightarrow \quad \{I\ \mathsf{nil}\}\ (\mathtt{iter}\, f\, l)\ \{\lambda\_.\ I\, l\}$$

Verification of iter

$$\begin{array}{rl} & \left(\forall x k.\ \{I\ k\}\ (f\ x)\ \{\lambda\_.\ I\ (k \& x)\}\right) \\ \Rightarrow & \{I\ \mathsf{nil}\}\ (\mathsf{iter}\ f\ l)\ \{\lambda\_.\ I\ l\} \end{array}$$

```
let rec iter f l =
  match l with
  | [] -> ()
  | x::t -> f x; iter f t
```

How to prove that the code satisfies its specification?



Verification of iter: generalized principle

Assume:

$$\forall x k.\ \{I\ k\}\ (f\ x)\ \{\lambda\_.\ I\ (k \& x)\}$$

Prove:

$$\{I\ \mathsf{nil}\}\ (\mathsf{iter}\ f\ l)\ \{\lambda\_.\ I\ l\}$$

Verification of iter: generalized principle

Assume:

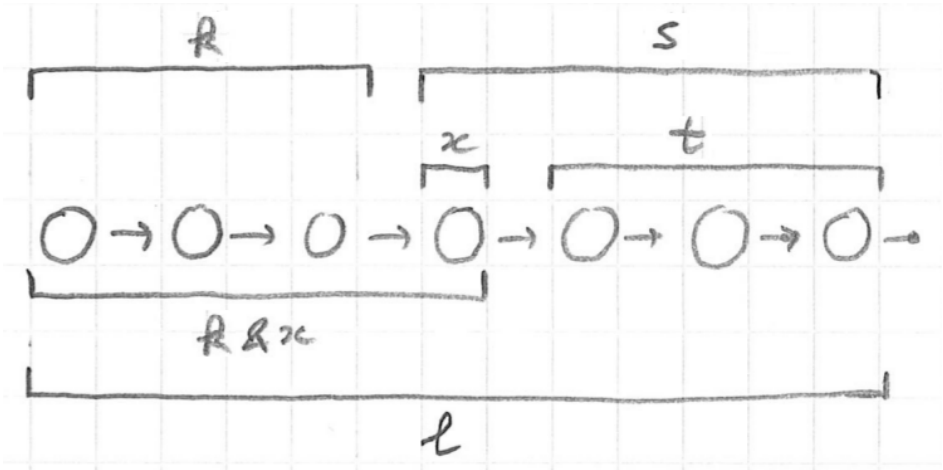
$$\forall x k.\ \{I\ k\}\ (f\ x)\ \{\lambda\_.\ I\ (k \& x)\}$$

Prove:

$$\{I\ \mathsf{nil}\}\ (\mathsf{iter}\ f\ l)\ \{\lambda\_.\ I\ l\}$$

Proof by induction over a generalized statement:

$$\forall s k.\quad \{I\ k\}\ (\mathsf{iter}\ f\ s)\ \{\lambda\_.\ I\ (k +\!\!+ s)\}$$



Verification of iter: induction

```
let rec iter f s =
  match s with
  | [] -> ()
  | x::t -> f x; iter f t
```

Assume: $\forall x k.\quad \{I\,k\}\ (f\,x)\ \{\lambda\_.\ I\,(k \& x)\}$

Prove: $\forall k s.\quad \{I\,k\}\ (\mathsf{iter}\,f\,s)\ \{\lambda\_.\ I\,(k +\!\!+ s)\}$

By induction on s:

- Case $s = \mathsf{nil}$. Goal is: $\{I\,k\}\ (\mathsf{iter}\,f\,\mathsf{nil})\ \{\lambda\_.\ I\,(k +\!\!+ \mathsf{nil})\}$.
  This triple simplifies to: $\{I\,k\}\ ()\ \{\lambda\_.\ I\,k\}$, which is correct.

Verification of iter: induction

```
let rec iter f s =
  match s with
  | [] -> ()
  | x::t -> f x; iter f t
```

Assume: $\forall xk.\quad \{I\,k\}\ (f\,x)\ \{\lambda\_.\ I\,(k \& x)\}$

Prove: $\forall ks.\quad \{I\,k\}\ (\mathsf{iter}\,f\,s)\ \{\lambda\_.\ I\,(k +\!\!+ s)\}$

By induction on s:

- Case $s = \mathsf{nil}$. Goal is: $\{I\,k\}\ (\mathsf{iter}\,f\,\mathsf{nil})\ \{\lambda\_.\ I\,(k +\!\!+ \mathsf{nil})\}$.
  This triple simplifies to: $\{I\,k\}\ ()\ \{\lambda\_.\ I\,k\}$, which is correct.
- Case $s = x :: t$. Goal is: $\{I\,k\}\ (\mathsf{iter}\,f\,(x :: t))\ \{\lambda\_.\ I\,(k +\!\!+ (x :: t))\}$.

$$\dfrac{\dfrac{}{\{I\,k\}\ (f\,x)\ \{\lambda\_.\ I\,(k \& x)\}}\ \text{HYPOTHESIS-ON-F} \qquad \dfrac{}{\{I\,(k \& x)\}\ (\mathsf{iter}\,f\,t)\ \{\lambda\_.\ I\,((k \& x) +\!\!+ t)\}}\ \text{INDUCTION-HYPOTHESIS}}{\{I\,k\}\ (f\,x;\ \mathsf{iter}\,f\,t)\ \{I\,((k \& x) +\!\!+ t)\}}\ \text{SEQ}$$

Invariant on remaining items

$$\big(\forall x k.\ \{I\ k\}\ (f\ x)\ \{\lambda\_.\ I\ (k \& x)\}\big) \quad \Rightarrow \quad \{I\ \mathsf{nil}\}\ (\mathsf{iter}\ f\ l)\ \{\lambda\_.\ I\ l\}$$

$$\big(\forall \ldots.\ \{\ldots\}\ (f\ x)\ \{\lambda\_.\ \ldots\}\big) \quad \Rightarrow \quad \{I'\ l\}\ (\mathsf{iter}\ f\ l)\ \{\lambda\_.\ I'\ \mathsf{nil}\}$$

Exercise:

- specify `iter` using an invariant that depends on the list of items remaining to process, instead of on the list of items already processed.
- prove the old specification derivable from the new one,
- prove the new specification derivable from the old (most general) one.

Invariant on remaining items

$$\big(\forall xk.\ \{I\,k\}\ (f\,x)\ \{\lambda\_.\ I\,(k\&x)\}\big) \quad\Rightarrow\quad \{I\ \mathsf{nil}\}\ (\mathsf{iter}\ f\ l)\ \{\lambda\_.\ I\,l\}$$

$$\big(\forall \ldots.\ \{\ldots\}\ (f\,x)\ \{\lambda\_.\ \ldots\}\big) \quad\Rightarrow\quad \{I'\,l\}\ (\mathsf{iter}\ f\ l)\ \{\lambda\_.\ I'\ \mathsf{nil}\}$$

Exercise:

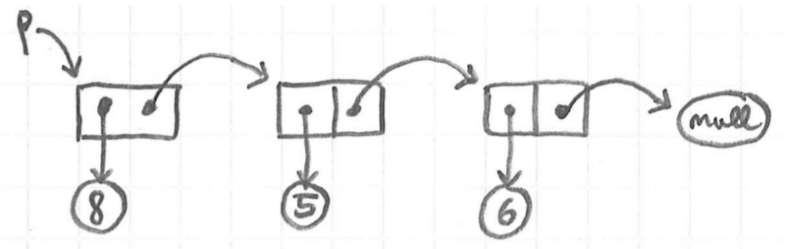
- specify `iter` using an invariant that depends on the list of items remaining to process, instead of on the list of items already processed.
- prove the old specification derivable from the new one,
- prove the new specification derivable from the old (most general) one.

$$\big(\forall xs.\ \{I'\,(x :: s)\}\ (f\,x)\ \{\lambda\_.\ I'\,s\}\big) \quad\Rightarrow\quad \{I'\,l\}\ (\mathsf{iter}\ f\ l)\ \{\lambda\_.\ I'\ \mathsf{nil}\}$$

$$I\,k \equiv \exists s.\ulcorner l = k +\!\!+ s\urcorner * I'\,s$$

$$I'\,s \equiv \exists k.\ulcorner l = k +\!\!+ s\urcorner * I\,k$$

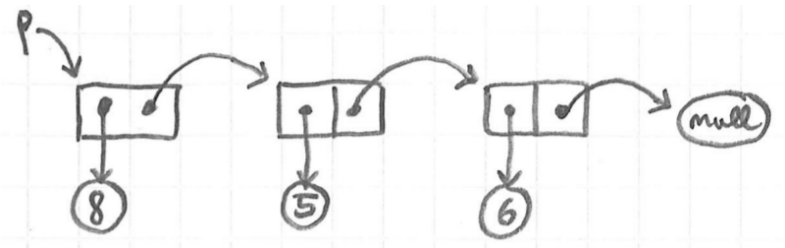
Iterating over a mutable list




```
let rec miter f p =
  if p == null
    then ()
    else (f p.hd; miter f p.tl)
```

Iterating over a mutable list

$$\forall f\, l\, I.\quad \big(\forall x\, k.\ \{I\ k\}\ (f\ x)\ \{\lambda\_.\ I\ (k \& x)\}\big)$$
$$\Rightarrow\quad \{I\ \mathsf{nil}\}\ (\mathsf{iter}\ f\ l)\ \{\lambda\_.\ I\ l\}$$



Specification:

$$\forall f\, p\, I\, l.\quad \big(\forall x\, k.\ \{I\ k\}\ (f\ x)\ \{\lambda\_.\ I\ (k \& x)\}\big)$$
$$\Rightarrow\quad \{p \rightsquigarrow \mathsf{MList}\ l\ *\ I\ \mathsf{nil}\}\ (\texttt{miter}\ f\ p)\ \{\lambda\_.\ p \rightsquigarrow \mathsf{MList}\ l\ *\ I\ l\}$$

Remark: calls to f will not modify the structure of the list while iterating.

Summary

Simplified:

$$\begin{array}{rl} & \left(\forall xk.\ \{I\,k\}\ (f\,x)\ \{\lambda\_.\ I\,(k \& x)\}\right) \\ \Rightarrow & \{I\,\mathsf{nil}\}\ (\mathsf{iter}\,f\,l)\ \{\lambda\_.\ I\,l\} \end{array}$$

Order-irrelevant:

$$\begin{array}{rl} & \left(\forall xk.\ x \in l \Rightarrow \{I\,k\}\ (f\,x)\ \{\lambda\_.\ I\,(k \& x)\}\right) \\ \Rightarrow & \{I\,\mathsf{nil}\}\ (\mathsf{iter}\,f\,l)\ \{\lambda\_.\ I\,l\} \end{array}$$

Most-general:

$$\begin{array}{rl} & \left(\forall xks.\ l = k +\!\!+ x :: s \Rightarrow \{I\,k\}\ (f\,x)\ \{\lambda\_.\ I\,(k \& x)\}\right) \\ \Rightarrow & \{I\,\mathsf{nil}\}\ (\mathsf{iter}\,f\,l)\ \{\lambda\_.\ I\,l\} \end{array}$$

Using remaining items (already most general):

$$\left(\forall xs.\ \{I'\,(x :: s)\}\ (f\,x)\ \{\lambda\_.\ I'\,s\}\right) \quad \Rightarrow \quad \{I'\,l\}\ (\mathsf{iter}\,f\,l)\ \{\lambda\_.\ I'\,\mathsf{nil}\}$$

Extension to mutable lists:

$$\begin{array}{rl} & \left(\forall xks.\ l = k +\!\!+ x :: s \Rightarrow \{I\,k\}\ (f\,x)\ \{\lambda\_.\ I\,(k \& x)\}\right) \\ \Rightarrow & \{p \rightsquigarrow \mathsf{MList}\,l\ *\ I\,\mathsf{nil}\}\ (\mathtt{miter}\,f\,p)\ \{\lambda\_.\ p \rightsquigarrow \mathsf{MList}\,l\ *\ I\,l\} \end{array}$$

Break?

Chapter 16

Other classic higher-order functions

Fold-left

```
let rec fold_left f a l =
  match l with
  | [] -> a
  | x::k -> fold_left f (f a x) k
```

Example:

$$\text{fold\_left}\ f\ a\ [6 :: 4 :: 7] \ = \ f\ (f\ (f\ a\ 6)\ 4)\ 7$$

Fold-left

```
let rec fold_left f a l =
  match l with
  | [] -> a
  | x::k -> fold_left f (f a x) k
```

Example:

$$\mathsf{fold\_left}\ f\ a\ [6 :: 4 :: 7] \quad = \quad f\,(f\,(f\,a\,6)\,4)\,7$$

Specification:

$$\begin{array}{lll} \forall f\,a\,l\,J. & & \big(\forall x\,i\,k.\ \{J\,i\,k\}\ (f\,i\,x)\ \{\lambda j.\ J\,j\,(k \& x)\}\big) \\ & \Rightarrow & \{J\,a\,\mathsf{nil}\}\ (\texttt{fold\_left}\ f\,a\,l)\ \{\lambda b.\ J\,b\,l\} \end{array}$$

Application of fold-left

$$\forall f\, a\, l\, J.\quad \big(\forall x\, i\, k.\ \{J\, i\, k\}\ (f\, i\, x)\ \{\lambda j.\ J\, j\, (k \& x)\}\big) \Rightarrow \{J\, a\, \mathsf{nil}\}\ (\texttt{fold\_left}\ f\, a\, l)\ \{\lambda b.\ J\, b\, l\}$$

```
let r = ref 0
let count_and_sum l =
  fold_left (fun a x -> incr r; a+x) 0 l
```

Exercise: give the instantiation of the invariant J in the code above.

Application of fold-left

$$\begin{array}{lll} \forall f\, a\, l\, J. & & \big(\forall x\, i\, k.\ \{J\, i\, k\}\ (f\, i\, x)\ \{\lambda j.\ J\, j\, (k \& x)\}\big) \\ & \Rightarrow & \{J\, a\, \mathsf{nil}\}\ (\texttt{fold\_left}\ f\, a\, l)\ \{\lambda b.\ J\, b\, l\} \end{array}$$

```
let r = ref 0
let count_and_sum l =
  fold_left (fun a x -> incr r; a+x) 0 l
```

Exercise: give the instantiation of the invariant J in the code above.

$$J\, i\, k \ \equiv\ (r \mapsto |k|) \ *\ \ulcorner i = \mathsf{Sum}\, k \urcorner$$

where $\mathsf{Sum}\, k \equiv \mathsf{Fold}\, (+)\, 0\, k$.

Fold-right

```
let rec fold_right f l a =
  match l with
  | [] -> a
  | x::k -> f x (fold_right f k a)
```

Example:

$$\text{fold\_right}\, f\, [6 :: 4 :: 7]\, a \;\equiv\; f\, 6\, (f\, 4\, (f\, 7\, a))$$

Exercise: give a specification for `fold_right`.

Fold-right

```
let rec fold_right f l a =
  match l with
  | [] -> a
  | x::k -> f x (fold_right f k a)
```

Example:

$$\text{fold\_right}\ f\ [6 :: 4 :: 7]\ a \ \equiv\ f\ 6\ (f\ 4\ (f\ 7\ a))$$

Exercise: give a specification for `fold_right`.

$$\begin{array}{lll} \forall f l a J. & & (\forall x i k.\ \{J\ i\ k\}\ (f\ x\ i)\ \{\lambda j.\ J\ j\ (x :: k)\}) \\ & \Rightarrow & \{J\ a\ \mathsf{nil}\}\ (\texttt{fold\_right}\ f\ l\ a)\ \{\lambda b.\ J\ b\ l\} \end{array}$$

Map: simple specification for pure functions

```
let rec map f l =
  match l with
  | [] -> []
  | x::k -> (f x)::(map f k)
```

Simple specification, for the case where `f` is pure:

$$\begin{aligned} \forall f\, l\, P. \quad & (\forall x.\ \{\ulcorner\urcorner\}\ (f\ x)\ \{\lambda x'.\ \ulcorner P\ x\ x'\urcorner\}) \\ \Rightarrow\ & \{\ulcorner\urcorner\}\ (\mathtt{map}\ f\ l)\ \{\lambda l'.\ \ulcorner \mathsf{Forall2}\ P\ l\ l'\urcorner\} \end{aligned}$$

where:

$$\frac{}{\mathsf{Forall2}\ P\ \mathsf{nil}\ \mathsf{nil}} \qquad \frac{P\ x\ x' \qquad \mathsf{Forall2}\ P\ l\ l'}{\mathsf{Forall2}\ P\ (x :: l)\ (x' :: l')}$$

Map: general specification

Specification of `map`:

$$\begin{array}{lll} \forall f\, l\, P. & & (\forall x.\ \{\ulcorner\urcorner\}\ (f\ x)\ \{\lambda x'.\ \ulcorner P\ x\ x'\urcorner\}) \\ & \Rightarrow & \{\ulcorner\urcorner\}\ (\mathsf{map}\ f\ l)\ \{\lambda l'.\ \ulcorner \mathsf{Forall2}\ P\ l\ l'\urcorner\} \end{array}$$

Specification of `iter`:

$$\begin{array}{lll} \forall f\, l\, I. & & \big(\forall x k.\ \{I\ k\}\ (f\ x)\ \{\lambda\_.\ I\ (k \mathbin{\&} x)\}\big) \\ & \Rightarrow & \{I\ \mathsf{nil}\}\ (\mathsf{iter}\ f\ l)\ \{\lambda\_.\ I\ l\} \end{array}$$

Map: general specification

Specification of `map`:

$$\begin{array}{rl} \forall f\, l\, P. & (\forall x.\ \{\ulcorner\urcorner\}\ (f\ x)\ \{\lambda x'.\ \ulcorner P\ x\ x'\urcorner\}) \\ \Rightarrow & \{\ulcorner\urcorner\}\ (\mathsf{map}\ f\ l)\ \{\lambda l'.\ \ulcorner \mathsf{Forall2}\ P\ l\ l'\urcorner\} \end{array}$$

Specification of `iter`:

$$\begin{array}{rl} \forall f\, l\, I. & \big(\forall x k.\ \{I\ k\}\ (f\ x)\ \{\lambda\_.\ I\ (k \& x)\}\big) \\ \Rightarrow & \{I\ \mathsf{nil}\}\ (\mathsf{iter}\ f\ l)\ \{\lambda\_.\ I\ l\} \end{array}$$

Combining the two:

$$\begin{array}{rl} \forall f\, l\, P\, I. & \big(\forall x k.\ \{I\ k\}\ (f\ x)\ \{\lambda x'.\ \ulcorner P\ x\ x'\urcorner * I\ (k \& x)\}\big) \\ \Rightarrow & \{I\ \mathsf{nil}\}\ (\mathsf{map}\ f\ l)\ \{\lambda l'.\ \ulcorner \mathsf{Forall2}\ P\ l\ l'\urcorner * I\ l\} \end{array}$$

Map: general specification, alternative

$$\begin{array}{lll} \forall f l P I. & & \left(\forall x k.\ \{I\ k\}\ (f\ x)\ \{\lambda x'.\ \ulcorner P\ x\ x' \urcorner * J\ (k \& x)\}\right) \\ & \Rightarrow & \{I\ \mathsf{nil}\}\ (\mathsf{map}\ f\ l)\ \{\lambda l'.\ \ulcorner \mathsf{Forall2}\ P\ l\ l' \urcorner * I\ l\} \end{array}$$

Alternative specification:

$$\begin{array}{lll} \forall f l J'. & & \left(\forall x k k'.\ \{J'\ k\ k'\}\ (f\ x)\ \{\lambda x'.\ J'\ (k \& x)\ (k' \& x')\}\right) \\ & \Rightarrow & \{J'\ \mathsf{nil}\ \mathsf{nil}\}\ (\mathsf{map}\ f\ l)\ \{\lambda l'.\ J'\ l\ l'\} \end{array}$$

Previous specification derivable from the above one:

$$J'\ k\ k' \ \equiv\ \ulcorner \mathsf{Forall2}\ P\ k\ k' \urcorner * I\ k$$

Sorting with comparison function

Example:

```
List.sort (fun x y -> x - y) [2;4;5;3;2;9]
```

Specification:

$$
\begin{array}{rl}
\forall f\, l.\ \forall (\preceq). & \\
& \mathsf{total\text{-}order}\ (\preceq) \\
\wedge & (\forall x y.\ \{\ulcorner\urcorner\}\ (f\ x\ y)\ \{\lambda n.\ulcorner n \leqslant 0 \Leftrightarrow x \preceq y\urcorner\}) \\
\Rightarrow & \{\ulcorner\urcorner\}\ (\mathtt{sort}\ f\ l)\ \{\lambda l'.\ \ulcorner \mathsf{permut}\ l\ l' \ \wedge\ \mathsf{sorted}\ (\preceq)\ l'\urcorner\}
\end{array}
$$

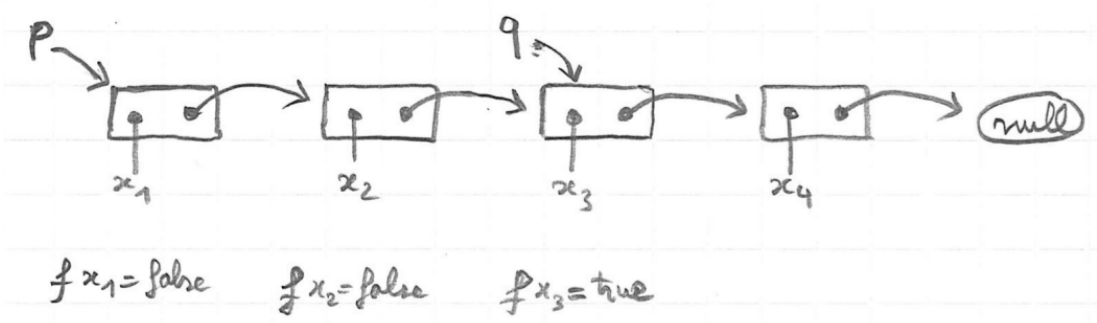
Find with a boolean predicate, on pure lists

```
let rec find f l =
  match l with
  | [] -> None
  | x::k -> if f x
            then Some x
            else find f k
```

Specification:

$$\forall f\, l\, P.\quad (\forall x.\ \{\ulcorner\urcorner\}\ (f\,x)\ \{\lambda b.\ \ulcorner b = \mathsf{true} \Leftrightarrow P\,x\urcorner\})$$
$$\Rightarrow\quad \{\ulcorner\urcorner\}\ (\texttt{find}\,f\,l)\ \{\lambda o.\ \ulcorner\ \mathsf{match}\,o\,\mathsf{with}\ulcorner\}$$
$$\mid \mathsf{None} \Rightarrow \mathsf{Forall}\,(\neg P)\,l$$
$$\mid \mathsf{Some}\,x \Rightarrow \exists k t.\ l = k +\!\!+ x :: t \wedge \mathsf{Forall}\,(\neg P)\,k \wedge P\,x$$

Find with a boolean predicate, on mutable lists

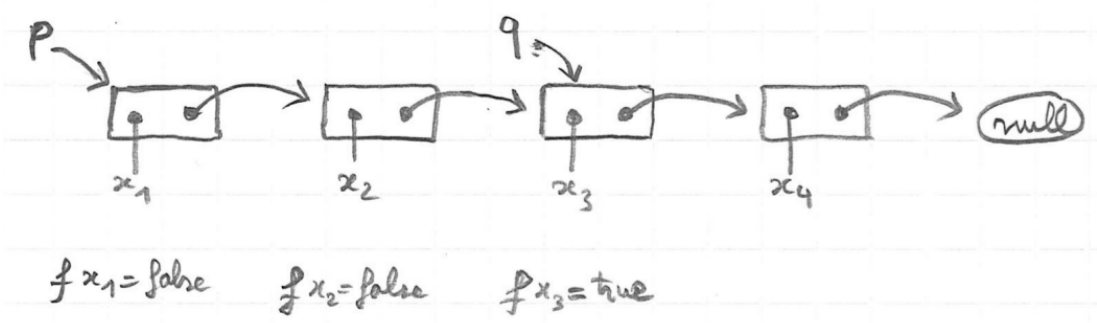


Specification:

$$\forall f\,p\,l\,P.\quad (\forall x.\ \{\ulcorner\urcorner\}\ (f\,x)\ \{\lambda b.\ulcorner b = \mathsf{true} \Leftrightarrow P\,x\urcorner\})$$

$$\Rightarrow\quad \{p \rightsquigarrow \mathsf{MList}\,l\}$$
$$(\texttt{mfind}\,f\,p)$$
$$\{\lambda o.$$

Find with a boolean predicate, on mutable lists

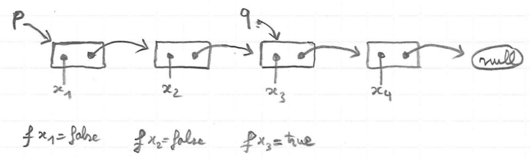


Specification:

$$\forall f\,p\,l\,P.\quad (\forall x.\ \{\ulcorner\urcorner\}\ (f\,x)\ \{\lambda b.\ulcorner b = \mathsf{true} \Leftrightarrow P\,x\urcorner\})$$

$$\Rightarrow\quad \{p \rightsquigarrow \mathsf{MList}\,l\}$$
$$(\texttt{mfind}\ f\ p)$$
$$\{\lambda o.\ \mathsf{match}\ o\ \mathsf{with}$$
$$\quad \mid \mathsf{None} \Rightarrow p \rightsquigarrow \mathsf{MList}\,l\ *\ \ulcorner\mathsf{Forall}\,(\neg P)\,l\urcorner$$

Find with a boolean predicate, on mutable lists



Specification:

$$
\begin{array}{ll}
\forall f p l P. & (\forall x.\ \{\ulcorner\urcorner\}\ (f\ x)\ \{\lambda b.\ \ulcorner b = \mathsf{true} \Leftrightarrow P\,x \urcorner\}) \\
\Rightarrow & \{p \rightsquigarrow \mathsf{MList}\, l\} \\
& (\texttt{mfind}\ f\ p) \\
& \{\lambda o.\ \mathsf{match}\ o\ \mathsf{with} \\
& \quad \mid \mathsf{None} \Rightarrow p \rightsquigarrow \mathsf{MList}\, l \ * \ \ulcorner \mathsf{Forall}\,(\neg P)\, l \urcorner \\
& \quad \mid \mathsf{Some}\, q \Rightarrow \exists k t.\ p \rightsquigarrow \mathsf{MlistSeg}\, q\, k \ * \ q \rightsquigarrow \mathsf{MList}\,(x :: t) \\
& \qquad\qquad * \ \ulcorner l = k +\!\!+ x :: t \ \wedge\ \mathsf{Forall}\,(\neg P)\, k \ \wedge\ P\, x \urcorner \quad \}
\end{array}
$$

Summary

$$
\begin{array}{ll}
 & \big(\forall x k.\ \{I\ k\}\ (f\ x)\ \{\lambda\_.\ I\ (k \& x)\}\big) \\
\Rightarrow & \{I\ \mathsf{nil}\}\ (\mathsf{iter}\ f\ l)\ \{\lambda\_.\ I\ l\}
\end{array}
$$

$$
\begin{array}{ll}
 & \big(\forall x i k.\ \{J\ i\ k\}\ (f\ i\ x)\ \{\lambda j.\ J\ j\ (k \& x)\}\big) \\
\Rightarrow & \{J\ a\ \mathsf{nil}\}\ (\mathsf{fold}\ f\ a\ l)\ \{\lambda b.\ J\ b\ l\}
\end{array}
$$

$$
\begin{array}{ll}
 & \big(\forall x k k'.\ \{J\ k\ k'\}\ (f\ x)\ \{\lambda x'.\ J\ (k \& x)\ (k' \& x')\}\big) \\
\Rightarrow & \{J\ \mathsf{nil}\ \mathsf{nil}\}\ (\mathsf{map}\ f\ l)\ \{\lambda l'.\ J\ l\ l'\}
\end{array}
$$

- Add the hypothesis $l = k +\!\!+ x :: s$ if the position of x matters.

- Boolean predicates: $\forall x.\ \{\ulcorner\urcorner\}\ (f\ x)\ \{\lambda b.\ \ulcorner b = \mathsf{true} \Leftrightarrow P\ x\urcorner\}$.
- Order functions: $\forall x y.\ \{\ulcorner\urcorner\}\ (f\ x\ y)\ \{\lambda n.\ \ulcorner n \leqslant 0 \Leftrightarrow x \preceq y\urcorner\}$.

Chapter 17

Higher-order representation predicates

Overview

1. Higher-order predicate:

$$p \rightsquigarrow \mathsf{MList}\, L \qquad \text{is generalized into} \qquad p \rightsquigarrow \mathsf{Mlistof}\, R\, L$$

2. Identity representation predicate:

$$p \rightsquigarrow \mathsf{Mlistof}\, \mathsf{Id}\, L \qquad \text{is the same as} \qquad p \rightsquigarrow \mathsf{MList}\, L$$

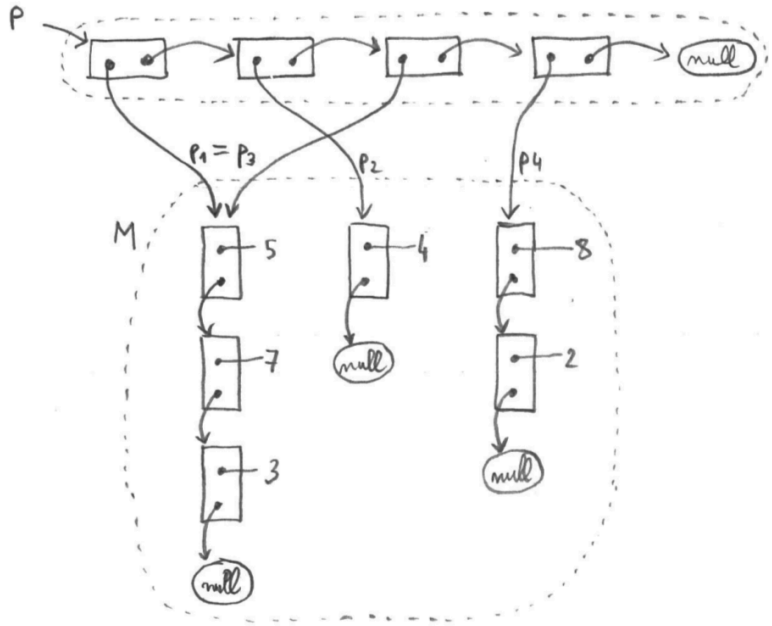
3. Control accesses:

$$\{p \rightsquigarrow \mathsf{MCellof}\, \mathsf{Id}\, v_1\, R_2\, V_2\}\ (p.\mathsf{hd})\ \{\lambda x.\ulcorner x = v_1 \urcorner * \ldots\}$$

4. Compose recursively:

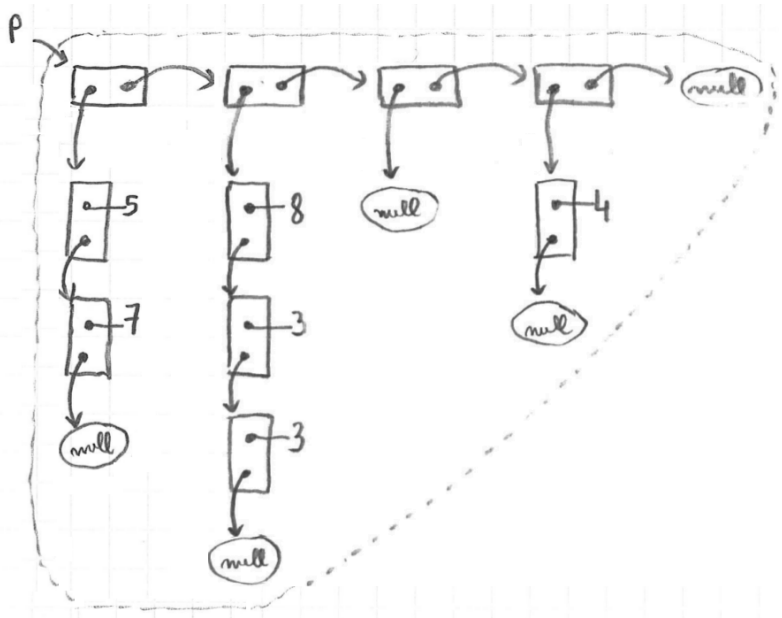
$$p \rightsquigarrow \mathsf{Nodeof}\, R\, X\, (\mathsf{Mlistof}\, (\mathsf{Narytreeof}\, R))\, L$$

Mutable list of possibly-aliased lists



$$p \rightsquigarrow \mathsf{MList}\, K \quad * \quad \left(\circledast_{(p_i, L_i) \in M} \; p_i \rightsquigarrow \mathsf{MList}\, L_i \right) \quad * \quad \ulcorner \forall p_i \in K.\ \ p_i \in \mathsf{dom}\, M \urcorner$$

Mutable list of disjoint mutable lists

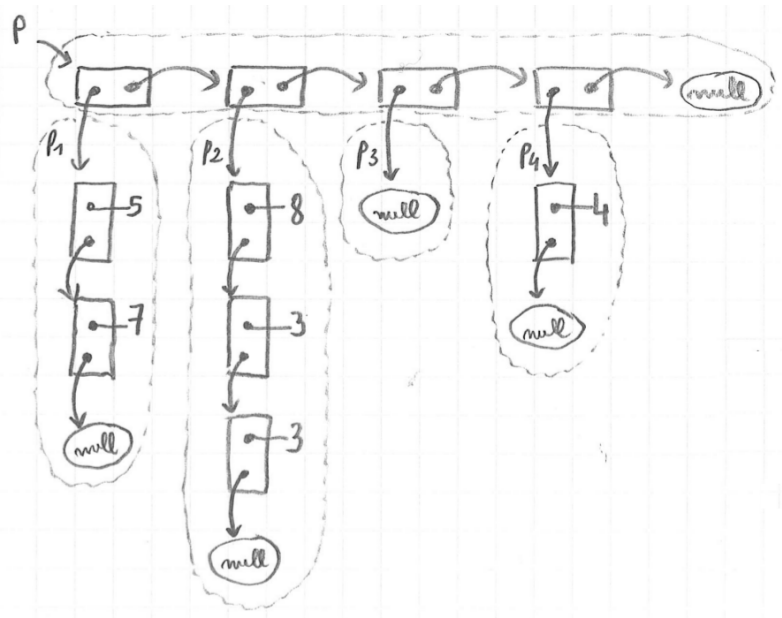


$$L \;=\; (5 :: 7 :: \mathsf{nil}) :: (8 :: 3 :: 3 :: \mathsf{nil}) \\ :: (\mathsf{nil}) :: (4 :: \mathsf{nil}) \;::\; \mathsf{nil}$$

$$p \rightsquigarrow \mathsf{MlistofMlist}\; L$$

(to be later generalized into: $p \rightsquigarrow \mathsf{Mlistof}\; R\; L$)

Representation using iterated star

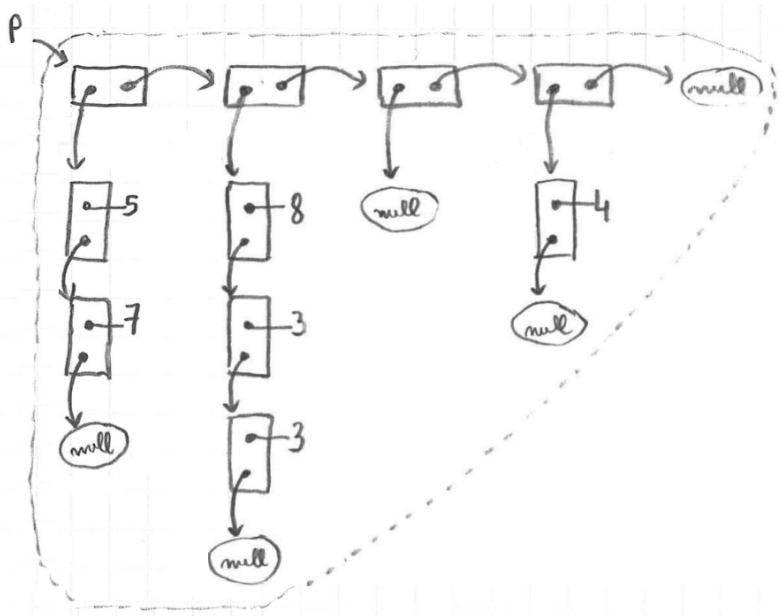


$$L \;=\; (5 :: 7 :: \mathsf{nil}) :: (8 :: 3 :: 3 :: \mathsf{nil}) :: (\mathsf{nil}) :: (4 :: \mathsf{nil}) :: \mathsf{nil}$$

$$K \;=\; p_1 :: p_2 :: p_3 :: p_4 :: \mathsf{nil}$$

$$p \rightsquigarrow \mathsf{MlistofMlist}\, L \quad \equiv \quad \exists K.\quad p \rightsquigarrow \mathsf{MList}\, K$$
$$* \quad \circledast_{i \in [0, |L|)} \, (K[i]) \rightsquigarrow \mathsf{MList}\, (L[i])$$
$$* \quad \ulcorner |K| = |L| \urcorner$$

Representation using a recursive predicate



$$L \;=\; (5 :: 7 :: \mathsf{nil}) :: (8 :: 3 :: 3 :: \mathsf{nil}) :: (\mathsf{nil}) :: (4 :: \mathsf{nil}) :: \mathsf{nil}$$

$$\begin{array}{lcl} p \rightsquigarrow \mathsf{MlistofMlist}\, L & \equiv & \mathsf{match}\, L\, \mathsf{with} \\ & & \mid \mathsf{nil} \Rightarrow \ulcorner p = \mathsf{null} \urcorner \\ & & \mid X :: L' \Rightarrow \exists x p'.\;\; p \rightsquigarrow \{\!|\mathsf{hd}{=}x;\; \mathsf{tl}{=}p'|\!\} \\ & & \qquad * \;\; p' \rightsquigarrow \mathsf{MlistofMlist}\, L' \\ & & \qquad * \;\; x \rightsquigarrow \mathsf{MList}\, X \end{array}$$

Generalization to a higher-order predicate

$$
\begin{array}{l}
p \rightsquigarrow \mathsf{MlistofMlist}\ L \equiv \mathsf{match}\ L\ \mathsf{with} \\
\quad \mid \mathsf{nil} \Rightarrow \ulcorner p = \mathsf{null} \urcorner \\
\quad \mid X :: L' \Rightarrow \exists x p'.\ \ p \rightsquigarrow \{\!|\mathsf{hd}{=}x;\ \mathsf{tl}{=}p'|\!\} \\
\qquad\qquad * \ \ p' \rightsquigarrow \mathsf{MlistofMlist}\ L' \\
\qquad\qquad * \ \ x \rightsquigarrow \mathsf{MList}\ X
\end{array}
$$

Generalization:

$$
\begin{array}{l}
p \rightsquigarrow \mathsf{Mlistof}\ R\ L \equiv \mathsf{match}\ L\ \mathsf{with} \\
\quad \mid \mathsf{nil} \Rightarrow \ulcorner p = \mathsf{null} \urcorner \\
\quad \mid X :: L' \Rightarrow \exists x p'.\ \ p \rightsquigarrow \{\!|\mathsf{hd}{=}x;\ \mathsf{tl}{=}p'|\!\} \\
\qquad\qquad * \ \ p' \rightsquigarrow \mathsf{Mlistof}\ R\ L' \\
\qquad\qquad * \ \ x \rightsquigarrow R\ X
\end{array}
$$

In particular:

$$
p \rightsquigarrow \mathsf{MlistofMlist}\ L \ = \ p \rightsquigarrow \mathsf{Mlistof}\ \mathsf{MList}\ L
$$

Type-checking

$p \rightsquigarrow \mathsf{Mlistof}\, R\, L$ *is a notation for* $\mathsf{Mlistof}\, R\, L\, p$ (of type Hprop)
$x \rightsquigarrow R\, X$ *is a notation for* $R\, X\, x$ (of type Hprop)

$$
\begin{array}{lcl}
p \rightsquigarrow \mathsf{Mlistof}\, R\, L & \equiv & \mathsf{match}\, L\, \mathsf{with} \\
 & & \mid \mathsf{nil} \Rightarrow \ulcorner p = \mathsf{null} \urcorner \\
 & & \mid X :: L' \Rightarrow \exists x p'. \quad p \rightsquigarrow \{\!|\mathsf{hd}{=}x;\ \mathsf{tl}{=}p'|\!\} \\
 & & \qquad * \ p' \rightsquigarrow \mathsf{Mlistof}\, R\, L' \\
 & & \qquad * \ x \rightsquigarrow R\, X
\end{array}
$$

Exercise: since $(p : \mathsf{loc})$ and $(x : \mathsf{Val})$ and $(X : A)$ for some A, what is the type of R? What is the type of Mlistof?

Type-checking

$p \rightsquigarrow \mathsf{Mlistof}\, R\, L$ *is a notation for* $\mathsf{Mlistof}\, R\, L\, p$ (of type Hprop)
$x \rightsquigarrow R\, X$ *is a notation for* $R\, X\, x$ (of type Hprop)

$$
\begin{array}{lcl}
p \rightsquigarrow \mathsf{Mlistof}\, R\, L & \equiv & \mathsf{match}\, L\, \mathsf{with} \\
 & & \mid \mathsf{nil} \Rightarrow \ulcorner p = \mathsf{null} \urcorner \\
 & & \mid X :: L' \Rightarrow \exists x p'. \quad p \rightsquigarrow \{\!|\mathsf{hd}{=}x;\ \mathsf{tl}{=}p'|\!\} \\
 & & \qquad * \ p' \rightsquigarrow \mathsf{Mlistof}\, R\, L' \\
 & & \qquad * \ x \rightsquigarrow R\, X
\end{array}
$$

Exercise: since $(p : \mathsf{loc})$ and $(x : \mathsf{Val})$ and $(X : A)$ for some A, what is the type of R? What is the type of Mlistof?

- $R : A \to \mathsf{Val} \to \mathsf{Hprop}$
- $\mathsf{Mlistof} : \forall A.\ (A \to \mathsf{Val} \to \mathsf{Hprop}) \to \mathsf{list} A \to \mathsf{loc} \to \mathsf{Hprop}$

The identity representation predicate

$$
\begin{array}{rcl}
p \rightsquigarrow \mathsf{Mlistof}\, R\, L & \equiv & \mathsf{match}\, L\, \mathsf{with} \\
& & \mid \mathsf{nil} \Rightarrow \ulcorner p = \mathsf{null} \urcorner \\
& & \mid X :: L' \Rightarrow \exists x p'.\quad p \rightsquigarrow \{\!|\mathsf{hd}{=}x;\ \mathsf{tl}{=}p'|\!\} \\
& & \qquad\qquad * \;\; p' \rightsquigarrow \mathsf{Mlistof}\, R\, L' \\
& & \qquad\qquad * \;\; x \rightsquigarrow R\, X
\end{array}
$$

$$
\begin{array}{rcl}
p \rightsquigarrow \mathsf{MList}\, L & \equiv & \mathsf{match}\, L\, \mathsf{with} \\
& & \mid \mathsf{nil} \Rightarrow \ulcorner p = \mathsf{null} \urcorner \\
& & \mid x :: L' \Rightarrow \exists p'.\quad p \rightsquigarrow \{\!|\mathsf{hd}{=}x;\ \mathsf{tl}{=}p'|\!\} \\
& & \qquad\qquad * \;\; p' \rightsquigarrow \mathsf{MList}\, L'
\end{array}
$$

Exercise: define the identity representation predicate Id such that

$$p \rightsquigarrow \mathsf{Mlistof}\, \mathsf{Id}\, L \;=\; p \rightsquigarrow \mathsf{MList}\, L$$

The identity representation predicate

$$
\begin{array}{rcl}
p \rightsquigarrow \mathsf{Mlistof}\, R\, L & \equiv & \mathsf{match}\, L\, \mathsf{with} \\
& & \mid \mathsf{nil} \Rightarrow \ulcorner p = \mathsf{null} \urcorner \\
& & \mid X :: L' \Rightarrow \exists x p'.\quad p \rightsquigarrow \{\!|\mathsf{hd}{=}x;\ \mathsf{tl}{=}p'|\!\} \\
& & \qquad\qquad * \;\; p' \rightsquigarrow \mathsf{Mlistof}\, R\, L' \\
& & \qquad\qquad * \;\; x \rightsquigarrow R\, X
\end{array}
$$

$$
\begin{array}{rcl}
p \rightsquigarrow \mathsf{MList}\, L & \equiv & \mathsf{match}\, L\, \mathsf{with} \\
& & \mid \mathsf{nil} \Rightarrow \ulcorner p = \mathsf{null} \urcorner \\
& & \mid x :: L' \Rightarrow \exists p'.\quad p \rightsquigarrow \{\!|\mathsf{hd}{=}x;\ \mathsf{tl}{=}p'|\!\} \\
& & \qquad\qquad * \;\; p' \rightsquigarrow \mathsf{MList}\, L'
\end{array}
$$

Exercise: define the identity representation predicate Id such that

$$p \rightsquigarrow \mathsf{Mlistof}\, \mathsf{Id}\, L \;=\; p \rightsquigarrow \mathsf{MList}\, L$$

Definition:

$$x \rightsquigarrow \mathsf{Id}\, X \;\equiv\; \ulcorner x = X \urcorner$$

Summary

1. Higher-order predicate:

$$p \rightsquigarrow \mathsf{MList}\, L \qquad \text{is generalized into} \qquad p \rightsquigarrow \mathsf{Mlistof}\, R\, L$$

2. Identity representation predicate:

$$p \rightsquigarrow \mathsf{Mlistof}\, \mathsf{Id}\, L \qquad \text{is the same as} \qquad p \rightsquigarrow \mathsf{MList}\, L$$

Chapter 18

Separating implication

Separating implication or "magic wand"

Recalling separating conjunction:

$$(P_1 * P_2)h \quad \equiv \quad \exists h_1, h_2.\ h = h_1 \uplus h_2 \wedge P_1\, h_1 \wedge P_2\, h_2$$

Introducing *separating implication*:

$$(P \mathrel{-\!\!*} Q)h \quad \equiv \quad \forall h_1.\ h \perp h_1 \wedge P\, h_1 \Rightarrow Q(h \uplus h_1)$$

Intuition:

$$(P \mathrel{-\!\!*} Q) * P \quad \rhd \quad Q$$

Rules:

$$\frac{R * P \vdash Q}{R \vdash (P \mathrel{-\!\!*} Q)} \qquad \frac{R_1 \vdash (P \mathrel{-\!\!*} Q) \qquad R_2 \vdash P}{R_1 * R_2 \vdash Q}$$

Separating implication examples

Exercise: Give heaps satisfying the following predicates:

1. $\ulcorner\urcorner \mathrel{-\!\!*} (1 \mapsto 2)$
2. $\ulcorner \mathsf{False} \urcorner \mathrel{-\!\!*} (1 \mapsto 2)$
3. $\ulcorner x \geqslant 1 \urcorner \mathrel{-\!\!*} \ulcorner x \geqslant 0 \urcorner$
4. $(1 \mapsto 4) \mathrel{-\!\!*} (1 \mapsto 4) * (2 \mapsto 3)$
5. $(1 \mapsto 2) \mathrel{-\!\!*} (1 \mapsto 2)$
6. $(1 \mapsto 2) \mathrel{-\!\!*} \ulcorner \mathsf{False} \urcorner$
7. $(1 \mapsto 2) \mathrel{-\!\!*} \ulcorner\urcorner$
8. $(1 \mapsto 2) \mathrel{-\!\!*} (1 \mapsto 3)$

Separating implication examples

Exercise: Give heaps satisfying the following predicates:

1. $\ulcorner\urcorner \mathrel{-\!\!*} (1 \mapsto 2)$ $\{(1, 2)\}$
2. $\ulcorner\text{False}\urcorner \mathrel{-\!\!*} (1 \mapsto 2)$
3. $\ulcorner x \geqslant 1\urcorner \mathrel{-\!\!*} \ulcorner x \geqslant 0\urcorner$
4. $(1 \mapsto 4) \mathrel{-\!\!*} (1 \mapsto 4) * (2 \mapsto 3)$
5. $(1 \mapsto 2) \mathrel{-\!\!*} (1 \mapsto 2)$
6. $(1 \mapsto 2) \mathrel{-\!\!*} \ulcorner\text{False}\urcorner$
7. $(1 \mapsto 2) \mathrel{-\!\!*} \ulcorner\urcorner$
8. $(1 \mapsto 2) \mathrel{-\!\!*} (1 \mapsto 3)$

Separating implication examples

Exercise: Give heaps satisfying the following predicates:

1. $\ulcorner\urcorner \mathrel{-\!\!*} (1 \mapsto 2)$ — $\{(1,2)\}$
2. $\ulcorner\text{False}\urcorner \mathrel{-\!\!*} (1 \mapsto 2)$ — all heaps
3. $\ulcorner x \geqslant 1\urcorner \mathrel{-\!\!*} \ulcorner x \geqslant 0\urcorner$
4. $(1 \mapsto 4) \mathrel{-\!\!*} (1 \mapsto 4) * (2 \mapsto 3)$
5. $(1 \mapsto 2) \mathrel{-\!\!*} (1 \mapsto 2)$
6. $(1 \mapsto 2) \mathrel{-\!\!*} \ulcorner\text{False}\urcorner$
7. $(1 \mapsto 2) \mathrel{-\!\!*} \ulcorner\urcorner$
8. $(1 \mapsto 2) \mathrel{-\!\!*} (1 \mapsto 3)$

Separating implication examples

Exercise: Give heaps satisfying the following predicates:

1. $\ulcorner\urcorner \mathrel{-\!\!*} (1 \mapsto 2)$ — $\{(1, 2)\}$
2. $\ulcorner \mathsf{False} \urcorner \mathrel{-\!\!*} (1 \mapsto 2)$ — all heaps
3. $\ulcorner x \geqslant 1 \urcorner \mathrel{-\!\!*} \ulcorner x \geqslant 0 \urcorner$ — only $\varnothing$
4. $(1 \mapsto 4) \mathrel{-\!\!*} (1 \mapsto 4) * (2 \mapsto 3)$
5. $(1 \mapsto 2) \mathrel{-\!\!*} (1 \mapsto 2)$
6. $(1 \mapsto 2) \mathrel{-\!\!*} \ulcorner \mathsf{False} \urcorner$
7. $(1 \mapsto 2) \mathrel{-\!\!*} \ulcorner\urcorner$
8. $(1 \mapsto 2) \mathrel{-\!\!*} (1 \mapsto 3)$

Separating implication examples

Exercise: Give heaps satisfying the following predicates:

1. $\ulcorner\urcorner \mathrel{-\!\!*} (1 \mapsto 2)$	$\{(1,2)\}$
2. $\ulcorner\mathsf{False}\urcorner \mathrel{-\!\!*} (1 \mapsto 2)$	all heaps
3. $\ulcorner x \geqslant 1\urcorner \mathrel{-\!\!*} \ulcorner x \geqslant 0\urcorner$	only $\varnothing$
4. $(1 \mapsto 4) \mathrel{-\!\!*} (1 \mapsto 4) * (2 \mapsto 3)$	$\{(2,3)\}$ and any h with $1 \in \mathsf{dom}(h)$
5. $(1 \mapsto 2) \mathrel{-\!\!*} (1 \mapsto 2)$	
6. $(1 \mapsto 2) \mathrel{-\!\!*} \ulcorner\mathsf{False}\urcorner$	
7. $(1 \mapsto 2) \mathrel{-\!\!*} \ulcorner\urcorner$	
8. $(1 \mapsto 2) \mathrel{-\!\!*} (1 \mapsto 3)$	

Separating implication examples

Exercise: Give heaps satisfying the following predicates:

1. $\ulcorner\urcorner \mathrel{-\!\!*} (1 \mapsto 2)$	$\{(1,2)\}$
2. $\ulcorner\mathsf{False}\urcorner \mathrel{-\!\!*} (1 \mapsto 2)$	all heaps
3. $\ulcorner x \geqslant 1\urcorner \mathrel{-\!\!*} \ulcorner x \geqslant 0\urcorner$	only $\varnothing$
4. $(1 \mapsto 4) \mathrel{-\!\!*} (1 \mapsto 4) * (2 \mapsto 3)$	$\{(2,3)\}$ and any h with $1 \in \mathsf{dom}(h)$
5. $(1 \mapsto 2) \mathrel{-\!\!*} (1 \mapsto 2)$	$\varnothing$ and any h with $1 \in \mathsf{dom}(h)$
6. $(1 \mapsto 2) \mathrel{-\!\!*} \ulcorner\mathsf{False}\urcorner$	
7. $(1 \mapsto 2) \mathrel{-\!\!*} \ulcorner\urcorner$	
8. $(1 \mapsto 2) \mathrel{-\!\!*} (1 \mapsto 3)$	

Separating implication examples

Exercise: Give heaps satisfying the following predicates:

1. $\ulcorner\urcorner \mathrel{-\!\!*} (1 \mapsto 2)$	$\{(1,2)\}$
2. $\ulcorner\mathsf{False}\urcorner \mathrel{-\!\!*} (1 \mapsto 2)$	all heaps
3. $\ulcorner x \geqslant 1\urcorner \mathrel{-\!\!*} \ulcorner x \geqslant 0\urcorner$	only $\varnothing$
4. $(1 \mapsto 4) \mathrel{-\!\!*} (1 \mapsto 4) * (2 \mapsto 3)$	$\{(2,3)\}$ and any h with $1 \in \mathsf{dom}(h)$
5. $(1 \mapsto 2) \mathrel{-\!\!*} (1 \mapsto 2)$	$\varnothing$ and any h with $1 \in \mathsf{dom}(h)$
6. $(1 \mapsto 2) \mathrel{-\!\!*} \ulcorner\mathsf{False}\urcorner$	any h with $1 \in \mathsf{dom}(h)$
7. $(1 \mapsto 2) \mathrel{-\!\!*} \ulcorner\urcorner$	
8. $(1 \mapsto 2) \mathrel{-\!\!*} (1 \mapsto 3)$	

Separating implication examples

Exercise: Give heaps satisfying the following predicates:

1. $\ulcorner\urcorner \mathrel{-\!\!*} (1 \mapsto 2)$	$\{(1,2)\}$
2. $\ulcorner\mathsf{False}\urcorner \mathrel{-\!\!*} (1 \mapsto 2)$	all heaps
3. $\ulcorner x \geqslant 1\urcorner \mathrel{-\!\!*} \ulcorner x \geqslant 0\urcorner$	only $\varnothing$
4. $(1 \mapsto 4) \mathrel{-\!\!*} (1 \mapsto 4) * (2 \mapsto 3)$	$\{(2,3)\}$ and any h with $1 \in \mathsf{dom}(h)$
5. $(1 \mapsto 2) \mathrel{-\!\!*} (1 \mapsto 2)$	$\varnothing$ and any h with $1 \in \mathsf{dom}(h)$
6. $(1 \mapsto 2) \mathrel{-\!\!*} \ulcorner\mathsf{False}\urcorner$	any h with $1 \in \mathsf{dom}(h)$
7. $(1 \mapsto 2) \mathrel{-\!\!*} \ulcorner\urcorner$	any h with $1 \in \mathsf{dom}(h)$
8. $(1 \mapsto 2) \mathrel{-\!\!*} (1 \mapsto 3)$	

Separating implication examples

Exercise: Give heaps satisfying the following predicates:

1. $\ulcorner\urcorner \mathrel{-\!\!*} (1 \mapsto 2)$	$\{(1,2)\}$
2. $\ulcorner\mathsf{False}\urcorner \mathrel{-\!\!*} (1 \mapsto 2)$	all heaps
3. $\ulcorner x \geqslant 1\urcorner \mathrel{-\!\!*} \ulcorner x \geqslant 0\urcorner$	only $\varnothing$
4. $(1 \mapsto 4) \mathrel{-\!\!*} (1 \mapsto 4) * (2 \mapsto 3)$	$\{(2,3)\}$ and any h with $1 \in \mathsf{dom}(h)$
5. $(1 \mapsto 2) \mathrel{-\!\!*} (1 \mapsto 2)$	$\varnothing$ and any h with $1 \in \mathsf{dom}(h)$
6. $(1 \mapsto 2) \mathrel{-\!\!*} \ulcorner\mathsf{False}\urcorner$	any h with $1 \in \mathsf{dom}(h)$
7. $(1 \mapsto 2) \mathrel{-\!\!*} \ulcorner\urcorner$	any h with $1 \in \mathsf{dom}(h)$
8. $(1 \mapsto 2) \mathrel{-\!\!*} (1 \mapsto 3)$	any h with $1 \in \mathsf{dom}(h)$

Separating implication examples

Exercise: Among the following heap entailments, which hold?

1. $P \rhd (Q \mathrel{-\!\!*} P * Q)$
2. $(Q \mathrel{-\!\!*} P * Q) \rhd P$
3. $(1 \mapsto 2) \mathrel{-\!\!*} (1 \mapsto 3) \rhd \ulcorner\mathsf{False}\urcorner$
4. $(1 \mapsto 2) \mathrel{-\!\!*} (1 \mapsto 2 * 2 \mapsto 8) \rhd 2 \mapsto 8$
5. $\ulcorner\urcorner \mathrel{-\!\!*} P \rhd P$
6. $P \rhd \ulcorner\urcorner \mathrel{-\!\!*} P$
7. $\ulcorner\urcorner \rhd (P \mathrel{-\!\!*} Q \mathrel{-\!\!*} P * Q)$
8. $\ulcorner P \rhd Q\urcorner \rhd (P \mathrel{-\!\!*} Q)$
9. $(P \mathrel{-\!\!*} Q) \rhd \ulcorner P \rhd Q\urcorner$

Separating implication examples

Exercise: Among the following heap entailments, which hold?

1. $P \rhd (Q \mathrel{-\!\!*} P * Q)$ yes: unfold and behold the definition of $*$
2. $(Q \mathrel{-\!\!*} P * Q) \rhd P$
3. $(1 \mapsto 2) \mathrel{-\!\!*} (1 \mapsto 3) \rhd \ulcorner\mathsf{False}\urcorner$
4. $(1 \mapsto 2) \mathrel{-\!\!*} (1 \mapsto 2 * 2 \mapsto 8) \rhd 2 \mapsto 8$
5. $\ulcorner\urcorner \mathrel{-\!\!*} P \rhd P$
6. $P \rhd \ulcorner\urcorner \mathrel{-\!\!*} P$
7. $\ulcorner\urcorner \rhd (P \mathrel{-\!\!*} Q \mathrel{-\!\!*} P * Q)$
8. $\ulcorner P \rhd Q\urcorner \rhd (P \mathrel{-\!\!*} Q)$
9. $(P \mathrel{-\!\!*} Q) \rhd \ulcorner P \rhd Q\urcorner$

Separating implication examples

Exercise: Among the following heap entailments, which hold?

1. $P \rhd (Q \mathrel{-\!\!*} P * Q)$ yes: unfold and behold the definition of $*$
2. $(Q \mathrel{-\!\!*} P * Q) \rhd P$ no e.g. with $P = Q = \ulcorner\text{False}\urcorner$
3. $(1 \mapsto 2) \mathrel{-\!\!*} (1 \mapsto 3) \rhd \ulcorner\text{False}\urcorner$
4. $(1 \mapsto 2) \mathrel{-\!\!*} (1 \mapsto 2 * 2 \mapsto 8) \rhd 2 \mapsto 8$
5. $\ulcorner\urcorner \mathrel{-\!\!*} P \rhd P$
6. $P \rhd \ulcorner\urcorner \mathrel{-\!\!*} P$
7. $\ulcorner\urcorner \rhd (P \mathrel{-\!\!*} Q \mathrel{-\!\!*} P * Q)$
8. $\ulcorner P \rhd Q\urcorner \rhd (P \mathrel{-\!\!*} Q)$
9. $(P \mathrel{-\!\!*} Q) \rhd \ulcorner P \rhd Q\urcorner$

Separating implication examples

Exercise: Among the following heap entailments, which hold?

1. $P \rhd (Q \mathrel{-\!\!*} P * Q)$ yes: unfold and behold the definition of $*$
2. $(Q \mathrel{-\!\!*} P * Q) \rhd P$ no e.g. with $P = Q = \ulcorner\text{False}\urcorner$
3. $(1 \mapsto 2) \mathrel{-\!\!*} (1 \mapsto 3) \rhd \ulcorner\text{False}\urcorner$ no e.g. $\{(1, 4)\}$ satisfies left
4. $(1 \mapsto 2) \mathrel{-\!\!*} (1 \mapsto 2 * 2 \mapsto 8) \rhd 2 \mapsto 8$
5. $\ulcorner\urcorner \mathrel{-\!\!*} P \rhd P$
6. $P \rhd \ulcorner\urcorner \mathrel{-\!\!*} P$
7. $\ulcorner\urcorner \rhd (P \mathrel{-\!\!*} Q \mathrel{-\!\!*} P * Q)$
8. $\ulcorner P \rhd Q\urcorner \rhd (P \mathrel{-\!\!*} Q)$
9. $(P \mathrel{-\!\!*} Q) \rhd \ulcorner P \rhd Q\urcorner$

Separating implication examples

Exercise: Among the following heap entailments, which hold?

1. $P \rhd (Q \mathrel{-\!\!*} P * Q)$ yes: unfold and behold the definition of $*$
2. $(Q \mathrel{-\!\!*} P * Q) \rhd P$ no e.g. with $P = Q = \ulcorner\text{False}\urcorner$
3. $(1 \mapsto 2) \mathrel{-\!\!*} (1 \mapsto 3) \rhd \ulcorner\text{False}\urcorner$ no e.g. $\{(1, 4)\}$ satisfies left
4. $(1 \mapsto 2) \mathrel{-\!\!*} (1 \mapsto 2 * 2 \mapsto 8) \rhd 2 \mapsto 8$ no (same)
5. $\ulcorner\urcorner \mathrel{-\!\!*} P \rhd P$
6. $P \rhd \ulcorner\urcorner \mathrel{-\!\!*} P$
7. $\ulcorner\urcorner \rhd (P \mathrel{-\!\!*} Q \mathrel{-\!\!*} P * Q)$
8. $\ulcorner P \rhd Q\urcorner \rhd (P \mathrel{-\!\!*} Q)$
9. $(P \mathrel{-\!\!*} Q) \rhd \ulcorner P \rhd Q\urcorner$

Separating implication examples

Exercise: Among the following heap entailments, which hold?

1. $P \rhd (Q \mathrel{-\!\!*} P * Q)$ yes: unfold and behold the definition of $*$
2. $(Q \mathrel{-\!\!*} P * Q) \rhd P$ no e.g. with $P = Q = \ulcorner\text{False}\urcorner$
3. $(1 \mapsto 2) \mathrel{-\!\!*} (1 \mapsto 3) \rhd \ulcorner\text{False}\urcorner$ no e.g. $\{(1, 4)\}$ satisfies left
4. $(1 \mapsto 2) \mathrel{-\!\!*} (1 \mapsto 2 * 2 \mapsto 8) \rhd 2 \mapsto 8$ no (same)
5. $\ulcorner\urcorner \mathrel{-\!\!*} P \rhd P$ yes, and...
6. $P \rhd \ulcorner\urcorner \mathrel{-\!\!*} P$
7. $\ulcorner\urcorner \rhd (P \mathrel{-\!\!*} Q \mathrel{-\!\!*} P * Q)$
8. $\ulcorner P \rhd Q\urcorner \rhd (P \mathrel{-\!\!*} Q)$
9. $(P \mathrel{-\!\!*} Q) \rhd \ulcorner P \rhd Q\urcorner$

Separating implication examples

Exercise: Among the following heap entailments, which hold?

1. $P \rhd (Q \mathrel{-\!\!*} P * Q)$ yes: unfold and behold the definition of $*$
2. $(Q \mathrel{-\!\!*} P * Q) \rhd P$ no e.g. with $P = Q = \ulcorner\text{False}\urcorner$
3. $(1 \mapsto 2) \mathrel{-\!\!*} (1 \mapsto 3) \rhd \ulcorner\text{False}\urcorner$ no e.g. $\{(1, 4)\}$ satisfies left
4. $(1 \mapsto 2) \mathrel{-\!\!*} (1 \mapsto 2 * 2 \mapsto 8) \rhd 2 \mapsto 8$ no (same)
5. $\ulcorner\urcorner \mathrel{-\!\!*} P \rhd P$ yes, and...
6. $P \rhd \ulcorner\urcorner \mathrel{-\!\!*} P$ ...yes: P and $\ulcorner\urcorner \mathrel{-\!\!*} P$ are equivalent
7. $\ulcorner\urcorner \rhd (P \mathrel{-\!\!*} Q \mathrel{-\!\!*} P * Q)$
8. $\ulcorner P \rhd Q\urcorner \rhd (P \mathrel{-\!\!*} Q)$
9. $(P \mathrel{-\!\!*} Q) \rhd \ulcorner P \rhd Q\urcorner$

Separating implication examples

Exercise: Among the following heap entailments, which hold?

1. $P \rhd (Q \mathrel{-\!\!*} P * Q)$ yes: unfold and behold the definition of $*$
2. $(Q \mathrel{-\!\!*} P * Q) \rhd P$ no e.g. with $P = Q = \ulcorner\text{False}\urcorner$
3. $(1 \mapsto 2) \mathrel{-\!\!*} (1 \mapsto 3) \rhd \ulcorner\text{False}\urcorner$ no e.g. $\{(1, 4)\}$ satisfies left
4. $(1 \mapsto 2) \mathrel{-\!\!*} (1 \mapsto 2 * 2 \mapsto 8) \rhd 2 \mapsto 8$ no (same)
5. $\ulcorner\urcorner \mathrel{-\!\!*} P \rhd P$ yes, and...
6. $P \rhd \ulcorner\urcorner \mathrel{-\!\!*} P$ ...yes: P and $\ulcorner\urcorner \mathrel{-\!\!*} P$ are equivalent
7. $\ulcorner\urcorner \rhd (P \mathrel{-\!\!*} Q \mathrel{-\!\!*} P * Q)$ yes: unfold $\rhd$ and obtain approx. (1)
8. $\ulcorner P \rhd Q\urcorner \rhd (P \mathrel{-\!\!*} Q)$
9. $(P \mathrel{-\!\!*} Q) \rhd \ulcorner P \rhd Q\urcorner$

Separating implication examples

Exercise: Among the following heap entailments, which hold?

1. $P \rhd (Q \mathrel{-\!\!*} P * Q)$ yes: unfold and behold the definition of $*$
2. $(Q \mathrel{-\!\!*} P * Q) \rhd P$ no e.g. with $P = Q = \ulcorner\text{False}\urcorner$
3. $(1 \mapsto 2) \mathrel{-\!\!*} (1 \mapsto 3) \rhd \ulcorner\text{False}\urcorner$ no e.g. $\{(1, 4)\}$ satisfies left
4. $(1 \mapsto 2) \mathrel{-\!\!*} (1 \mapsto 2 * 2 \mapsto 8) \rhd 2 \mapsto 8$ no (same)
5. $\ulcorner\urcorner \mathrel{-\!\!*} P \rhd P$ yes, and...
6. $P \rhd \ulcorner\urcorner \mathrel{-\!\!*} P$ ...yes: P and $\ulcorner\urcorner \mathrel{-\!\!*} P$ are equivalent
7. $\ulcorner\urcorner \rhd (P \mathrel{-\!\!*} Q \mathrel{-\!\!*} P * Q)$ yes: unfold $\rhd$ and obtain approx. (1)
8. $\ulcorner P \rhd Q\urcorner \rhd (P \mathrel{-\!\!*} Q)$ yes
9. $(P \mathrel{-\!\!*} Q) \rhd \ulcorner P \rhd Q\urcorner$

Separating implication examples

Exercise: Among the following heap entailments, which hold?

1. $P \rhd (Q \mathrel{-\!\!*} P * Q)$ yes: unfold and behold the definition of $*$
2. $(Q \mathrel{-\!\!*} P * Q) \rhd P$ no e.g. with $P = Q = \ulcorner\text{False}\urcorner$
3. $(1 \mapsto 2) \mathrel{-\!\!*} (1 \mapsto 3) \rhd \ulcorner\text{False}\urcorner$ no e.g. $\{(1, 4)\}$ satisfies left
4. $(1 \mapsto 2) \mathrel{-\!\!*} (1 \mapsto 2 * 2 \mapsto 8) \rhd 2 \mapsto 8$ no (same)
5. $\ulcorner\urcorner \mathrel{-\!\!*} P \rhd P$ yes, and...
6. $P \rhd \ulcorner\urcorner \mathrel{-\!\!*} P$ ...yes: P and $\ulcorner\urcorner \mathrel{-\!\!*} P$ are equivalent
7. $\ulcorner\urcorner \rhd (P \mathrel{-\!\!*} Q \mathrel{-\!\!*} P * Q)$ yes: unfold $\rhd$ and obtain approx. (1)
8. $\ulcorner P \rhd Q\urcorner \rhd (P \mathrel{-\!\!*} Q)$ yes
9. $(P \mathrel{-\!\!*} Q) \rhd \ulcorner P \rhd Q\urcorner$ no, e.g. $P = \ulcorner\urcorner$ and $Q = 1 \mapsto 2$