1 Ramsey's Theorem

Notations and Definitions.

- Given $A \subseteq \mathbb{N}$, we let $[A]^2 \subseteq A \times A$ be the set of all $(n, m) \in A \times A$ such that $n < m$.

- A set $U \subseteq \mathbb{N}$ is **unbounded** if for all $n \in \mathbb{N}$ there is $m \in U$ such that $n \leq m$.

- Given $k \in \mathbb{N}$, a $k$-coloring (of $[\mathbb{N}]^2$) is a function $C : [\mathbb{N}]^2 \to \{0, \ldots, k\}$.

A set $A \subseteq \mathbb{N}$ is **homogeneous** for $C$ if there is a color $i \in \{0, \ldots, k\}$ such that for all $(n, m) \in [A]^2$, we have $C(n, m) = i$.

The Infinite Ramsey’s Theorem is the following:

**Theorem 1.1** (Ramsey). *For all $k \in \mathbb{N}$ and all $k$-coloring $C$, there is an unbounded set which is homogeneous for $C$.***

The following property is essential to the proof of Theorem 1.1.

**Lemma 1.2** (Infinite Pigeonhole Principle). *Let $U \subseteq \mathbb{N}$ be unbounded and $(I_i)_{0 \leq i \leq k}$ be subsets of $\mathbb{N}$ such that $U \subseteq I_0 \cup \cdots \cup I_k$. There is an unbounded $V \subseteq U$ such that $V \subseteq I_i$ for some $i \in \{0, \ldots, k\}$.***

**Question 1.3.** Prove Lemma 1.2
**A Proof of Ramsey’s Theorem.** We propose to formalize the following proof of Theorem 1.1. Fix \( k \in \mathbb{N} \) and a \( k \)-coloring \( C \) of \([\mathbb{N}]^2\).

We define, by induction on \( n \in \mathbb{N} \), a sequence \( (U_n)_{n \in \mathbb{N}} \) of subsets of \( \mathbb{N} \) such that \( U_{n+1} \subseteq U_n \) and \( U_n \) is unbounded. Let \( U_0 := \mathbb{N} \). Let \( n \in \mathbb{N} \) and assume that \( U_n \) has been defined. Consider, for each color \( i \in \{0, \ldots, k\} \), the set \( V^i_n \subseteq U_n \) defined as follows:

\[
V^i_n := \{ p \mid p \in U_n \text{ and } p > m \text{ and } C(m, p) = i \}
\]

where \( m := \min(U_n) \)

Let \( U_{n+1} := V^i_n \) for some \( i \in \{0, \ldots, k\} \) such that

(i) \( V^i_n \) is unbounded,

(ii) and for all \( j \in \{0, \ldots, k\} \), if \( V^j_n \) is unbounded then \( \min(V^i_n) \leq \min(V^j_n) \).

**Question 1.4.** Show that the sequence \( (U_n)_{n \in \mathbb{N}} \) is well defined and unique.

In order to obtain Theorem 1.1, it remains to extract an homogeneous unbounded set from the sequence \( (\min(U_n))_{n \in \mathbb{N}} \).

**Question 1.5.** Show Theorem 1.1.

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### 2 Toward a Formalized Proof of Ramsey’s Theorem

Fix \( k \in \mathbb{N} \) and a \( k \)-coloring \( C \) of \([\mathbb{N}]^2\). In order to formalize the above proof in Coq (with classical logic), we propose to define the sequence \( (\min(U_n))_{n \in \mathbb{N}} \) as a predicate \( \text{Seq} : \mathbb{N} \rightarrow \text{Prop} \). We shall however not explicitly define the sequence \( (U_n)_{n \in \mathbb{N}} \).

Let

\[
\text{SU} := \{ \min(U_n) \mid n \in \mathbb{N} \}
\]

Given \( S \subseteq \mathbb{N} \) and \( a \in \mathbb{N} \), let \( W(S, a) \) be the set of all \( t \in \mathbb{N} \) such that for all \( b < a \) with \( b \in S \),

(i) \( C(b, a) = C(b, t) \),

(ii) and \( C(b, a) = C(b, c) \) for all \( c \in S \) such that \( b < c < a \).

**Question 2.1.** Show that for all \( n \in \mathbb{N} \), we have \( U_n \subseteq W(\text{SU}, \min(U_n)) \).

**Question 2.2.** Let \( a \in \mathbb{N} \) and consider \( S, T \subseteq \mathbb{N} \) such that

\[
\forall b < a \, (b \in S \iff b \in T)
\]

Show that \( W(S, a) = W(T, a) \).

**Question 2.3.** Show that there is a unique \( \text{Seq} \subseteq \mathbb{N} \) such that

\[
\forall a \in \mathbb{N} \, (a \in \text{Seq} \iff W(\text{Seq}, a) \text{ is unbounded})
\]  

(1)

The Coq predicate \( \text{Seq} : \mathbb{N} \rightarrow \text{Prop} \) will formalize the set \( \text{Seq} \). The interest of \( \text{Seq} \) is that it is actually equal to \( \text{SU} \), but makes no reference to the sequence \( (U_n)_{n \in \mathbb{N}} \).

**Theorem 2.4.** For all \( a \in \mathbb{N} \), we have

\[
a \in \text{SU} \iff a \in \text{Seq}
\]

Theorem 2.4 can be shown by well-founded induction.
Question 2.5. Let $a \in SU$ and assume that
\[ \forall b < a \ (b \in SU \iff b \in Seq) \] (IH)
Show that $a \in Seq$.

For the converse, it may be useful to use the following property.

Question 2.6. Let $a \in Seq$ and assume (IH). Show that for all $p \in \mathbb{N}$ such that $\min(U_p) < a$ and $a \in U_p$, we have $a \in U_{p+1}$.

Question 2.7. Show Theorem 2.4.

3 Coq Formalization

The formalization of the above proof of Ramsey’s theorem will be done in two steps:

1. To show Theorem 1.1 assuming a Coq predicate $Seq : \mathbb{N} \rightarrow \text{Prop}$ which formalizes $Seq$.

2. To define such a Coq predicate $Seq$ using well-founded induction on natural numbers and the accessibility predicate of Coq.

Step (2) is deferred until Section 3.2 below. For step (1), we use classical logic. In particular, we will need the Minimum Principle and the Infinite Pigeonhole Principle. The Minimum Principle also uses well-founded induction on $lt : \mathbb{N} \rightarrow \mathbb{N} \rightarrow \text{Prop}$ (most often used with the Coq notation $<$), which does not require classical logic.

Question 3.1 (Coq). Prove Theorem LtInd of Section WF_Induction.

Question 3.2 (Coq). Prove Theorem MinP of Section Minimum_Principle.

Question 3.3 (Coq). Prove Theorem IPP of Section IPP.

3.1 Formalization of the Proof Assuming $Seq$

The Coq variable $Seq : \mathbb{N} \rightarrow \text{Prop}$ of Section Ramsey, together with the hypothesis DefSeq, formalises a set $Seq$ satisfying condition (1) of Section 2.

We propose to first formally prove that $Seq$ is unbounded and then derive a formal proof of Ramsey’s Theorem.

An alternative proof that $Seq$ is unbounded. Theorem 2.4 already implies that $Seq$ is unbounded. However, we shall not formalize this proof since we do not formalise the sequence $(U_n)_{n \in \mathbb{N}}$. In order to show from scratch that $Seq$ is unbounded, we propose the following proof plan.

Given $n \in \mathbb{N}$, let $CumHom_n \subseteq \mathbb{N}$ be such that for all $a \in \mathbb{N}$, $a \in CumHom_n$ if and only if the following set $W'(n,a)$ is unbounded. Given $n, a \in \mathbb{N}$, the set $W'(n,a)$ is the set of all $t \in \mathbb{N}$ such that for all $b < n$ with $b \in Seq$,

(i) $C(b,a) = C(b,t)$,

(ii) and $C(b,a) = C(b,c)$ for all $c \in Seq$ such that $b < c < n$.

Question* 3.4. Let $n \in \mathbb{N}$.

(i) Show that for all $a \geq n$, if $a \in Seq$ then $a \in CumHom_n$.
(ii) Show that $a \in \text{Seq}$ if $a$ is the least element of $\text{CumHom}_n$ which is $\geq n$.

Hence, we get that $\text{Seq}$ is unbounded as soon as we prove that for all $n \in \mathbb{N}$, $\text{CumHom}_n$ contains some $a \geq n$. This can be shown using a third sequence of sets $(\text{Hom}_n)_{n \in \mathbb{N}}$, where $\text{Hom}_n$ is the set of all $a \in \mathbb{N}$ such that for all $b, c \in \text{Seq}$ such that $b < c < n$, we have $C(b, c) = C(b, a)$.

**Question*** 3.5. Let $n \in \mathbb{N}$.

(i) Show that $\text{Hom}_n$ is unbounded.

(ii) Show that there is an $a \geq n$ such that

$$\{ t \mid t \in \text{Hom}_n \text{ and } \forall b \in \text{Seq}(b < n \implies C(b, t) = C(b, a)) \} \text{ is unbounded}$$

(iii) Show that there is an $a \geq n$ such that $a \in \text{CumHom}_n$.

**The Coq predicate Seq is unbounded.** We begin by proving in Coq that for all $n \in \mathbb{N}$, there is an $a \in \text{CumHom}_n$ such that $a \geq n$. The sequences of sets $(\text{CumHom}_n)_{n \in \mathbb{N}}$ and $(\text{Hom}_n)_{n \in \mathbb{N}}$ are formalized by the predicates

$$\text{CumHom} : \mathbb{N} \to \mathbb{N} \to \text{Prop} \quad \text{and} \quad \text{Hom} : \mathbb{N} \to \mathbb{N} \to \text{Prop}$$

**Question*** 3.6 (Coq). Prove Lemma $\text{Hom}_\text{UB}$.

**Question*** 3.7 (Coq). Prove Lemma $\text{CumHom}_\text{Hom}$.

**Question*** 3.8 (Coq). Prove Lemma $\text{CumHom}_\text{UB}$.

From this, reasoning as above one can prove that $\text{Seq} : \mathbb{N} \to \text{Prop}$ is unbounded.

**Question*** 3.9 (Coq). Prove Lemma $\text{Seq}_\text{UB}$.

**Proof of Ramsey's Theorem.** We can now adapt the reasoning of Question 1.5 and derive a formal proof of Ramsey's Theorem.

**Question*** 3.10 (Coq). Prove Theorem $\text{Ramsey}_\text{MinPIPP EM}$.

### 3.2 Definition of Seq by Well-Founded Induction

Consider a type $A : \text{Type}$ and a relation $R : A \to A \to \text{Prop}$. The well-founded part of $R$ is called in Coq its accessible part. The Coq inductive predicate $\text{Acc} A R : A \to \text{Prop}$ provides a structural induction principle to reason by well-founded induction in the accessible part of $R$.

In particular, if $R$ is (provably in Coq) well-founded, then every $x : A$ is accessible. In this case, $\text{Acc} A R : A \to \text{Prop}$ allows to reason by well-founded induction on $R$.

**Question** 3.11 (Coq). Prove Lemma $\text{acc}$.

We define $\text{Seq} : \mathbb{N} \to \text{Prop}$ by well-founded induction on the strict ordering $<$ of $\mathbb{N}$ (recall that $<$ is a notation for $\mathbb{N} \to \mathbb{N} \to \text{Prop}$).

**Question** 3.12 (Coq). Complete the definition of

\[
\text{Fixpoint seq (x : nat) (a : Acc lt x) {struct a} : Prop}
\]
Given \( x : \text{nat} \), the predicate \( \text{seq} \ x \) formally depends on a proof \( a \) of \( \text{Acc} \ \text{lt} \ x \). The definition of \( \text{seq} \) makes sense because it is provably independent from the choice of a particular \( a : \text{Acc} \ \text{lt} \ x \).

**Question 3.13 (Coq).** Prove Lemma \( \text{seq}_\text{invariant} \).

Since we have

\[
\text{Lemma acc} : \forall x, \text{Acc} \ \text{lt} \ x
\]

it is natural to define \( \text{Seq} \) as

\[
\text{Definition Seq} \ x := \text{seq} \ x (\text{acc} \ x)
\]

It remains to prove that the Coq predicate \( \text{Seq} \) actually formalizes the set \( \text{Seq} \).

**Question 3.14 (Coq).** Prove Theorem \( \text{Seq}_\text{Correct} \).

### 3.3 Putting Everything Together

**Question 3.15 (Coq).** Prove Theorem \( \text{Ramsey} \).